1. The integral $\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx$ certainly converges. One simple way to try to approximate its value is to write it as the sum of two integrals:

$$\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx = \int_1^A \frac{e^{-5x}}{1+x^2}\,dx + \int_A^\infty \frac{e^{-5x}}{1+x^2}\,dx$$

where A is some large number. Then any of our numerical techniques could be used to approximate the first (proper!) definite integral on the right-hand side of the equation. We could just neglect the second integral if we knew that it was small.

a) Compare the second integral on the right to an integral over the same interval of a simpler function to try to find an A so that the improper integral to be dropped will have a value smaller than 10^{-5}.

b) Use a numerical technique to try to evaluate the first integral with an error less than $\pm 10^{-5}$. *You will definitely need a programmable calculator or a computer for this part!*

c) Compute $\int_1^\infty \frac{e^{-5x}}{1+x^2}\,dx$ with an error less than $\pm 10^{-4}$.

Note There is **no** unique or best possible answer to this question, either for the "simpler" function or for the A.

2. The following information is known about a function T and its derivatives:

x	$T(x)$	$T'(x)$	$T''(x)$
1	2	-2	2
2	3	6	5
3	7	4	-4
4	2	5	7

a) Compute $\int_2^3 T'(x)\,dx$.

b) Compute $\int_2^3 T''(x)\,dx$.

c) Compute $\int_2^3 x\,dx$.

d) Compute $\int_2^3 xT''(x)\,dx$.

Better not look at b) and c)! Integrate by parts.

e) Compute $\int_2^3 x^2T'''(x)\,dx$.

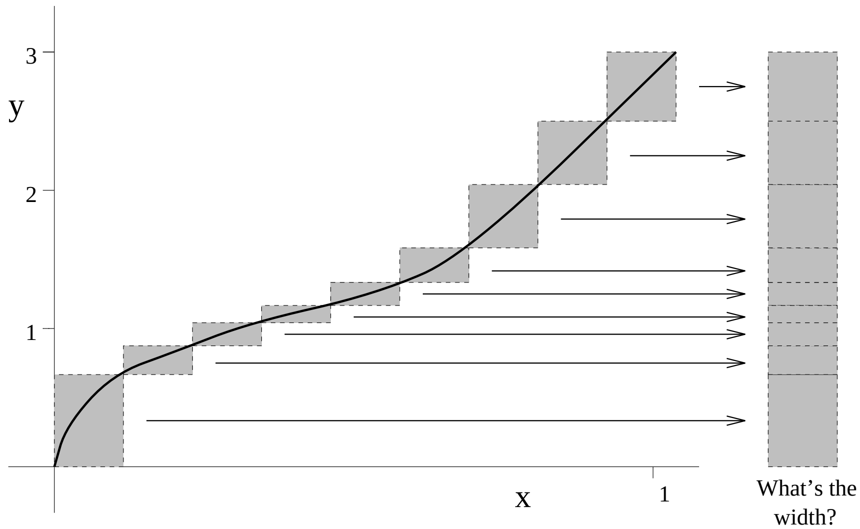
And again and again.

OVER

3. Suppose $f(x) = \sqrt{3x + 6x^4}$.

a) Prove that f is increasing on the interval $[0, 1]$.

b) Write down a finite sum which will be within 10^{-10} of the true value of $\int_0^1 f(x)\,dx$. You are *not* asked to actually compute the sum, just describe it in any convenient fashion. Hint: your reasoning and your explanation may be guided by the picture below:



The horizontal and vertical axes have different scales.

4. Many numerical methods for definite integrals arise by looking at what happens to simple polynomials.

a) Consider the equation $\int_0^1 f(x)\,dx = \frac{f(0)+f(1)}{2}$. Verify that if $f(x) = A + Bx$ then this equation is correct. If $f(x) = x^2$ then this equation is not correct.

This equation leads to the Trapezoidal Rule.

b) Consider the equation $\int_0^1 f(x)\,dx = f\left(\frac{1}{2}\right)$. Verify that if $f(x) = A + Bx$ then this equation is correct. If $f(x) = x^2$ then this equation is not correct.

This equation leads to the Midpoint Rule.

c) Consider the equation $\int_0^1 f(x)\,dx = \frac{1}{3}f(0) + \frac{2}{3}f\left(\frac{1}{3}\right)$. Verify that if $f(x) = A + Bx + Cx^2$ then this equation is correct. If $f(x) = x^3$ then this equation is not correct.

This equation leads to Radau Quadrature.

Moral Two function evaluations can get linear accuracy. One function evaluation can get linear accuracy. But two well-chosen function evaluations can get quadratic accuracy. If function evaluations are difficult to get (expensive or hard to measure), then ...