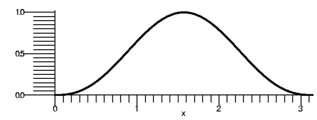
640:152:1, 2, & 3 **Answers to version A of the First Exam** **2/28/2008**

Here are answers that would earn full credit. Other methods may also be valid.

(12) 1. Verify that $\int_1^3 \frac{2x^2+7x+1}{x(x+1)^2}\,dx = 1+\ln(2)+\ln(3)$.

Answer We find A, B, and C so that $\frac{2x^2+7x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. Combine the fractions on the right-hand side of the equation. The result is a fraction with the same denominator as the left-hand side, so the tops must be equal: $2x^2+7x+1 = A(x+1)^2 + Bx(x+1) + Cx$. If $x=0$, we get $1=A$. If $x=-1$, we get $2-7+1=-C$ so $C=4$. The x^2 coefficients on both sides must be the same, so $2=A+B$ and therefore $B=1$. Evaluation: $\int_1^3 \left(\frac{1}{x}+\frac{1}{x+1}+\frac{4}{(x+1)^2}\right)dx = \ln(x)+\ln(x+1)-\frac{4}{x+1}\Big|_1^3 = \big(\ln(3)+\ln(4)-1\big)-\big(\ln(1)+\ln(2)-\frac{4}{2}\big) = \big(\ln(3)+2\ln(2)-1\big)-\big(0+\ln(2)-2\big) = 1+\ln(3)+\ln(2)$.

(14) 2. A region R in the plane has boundary $y=\big(\sin(x)\big)^{5/2}$ (a graph is shown to the right), the x-axis, $x=0$, and $x=\pi$.



a) Find the volume of the solid that results from rotating R around the x-axis.

Answer We compute $\pi\int_0^\pi (f(x))^2\,dx = \pi\int_0^\pi \big(\sin(x)\big)^5\,dx$. But $\int\big(\sin(x)\big)^5\,dx = \int\big(\sin(x)\big)^4\sin(x)dx = \int\Big(1-\big(\cos(x)\big)^2\Big)^2\sin(x)dx$. If $w=\cos(x)$ then $dw=-\sin(x)\,dx$ (note the minus!) so the integral becomes $-\int\big(1-w^2\big)^2\,dw = -\int 1-2w^2+w^4\,dw = -\big(w-\frac{2}{3}w^3+\frac{1}{5}w^5\big)+C = -\cos(x)+\frac{2}{3}(\cos(x))^3-\frac{1}{5}(\cos(x))^5+C$. Evaluation: $-\cos(x)+\frac{2}{3}(\cos(x))^3-\frac{1}{5}(\cos(x))^5\big|_0^\pi = (1-\frac{2}{3}+\frac{1}{5})-(-1+\frac{2}{3}-\frac{1}{5}) = \frac{16}{15}$. The volume is $\frac{16}{15}\pi$.

b) Write a definite integral for the volume of the solid that results from rotating R around the y-axis.

Answer $\int_0^\pi 2\pi x f(x)\,dx = \int_0^\pi 2\pi x\big(\sin(x)\big)^{5/2}\,dx$.

(12) 3. A full bucket of water initially weighs 11 pounds and drips at a constant rate. The bucket itself weighs 1 pound. As the bucket is raised 8 feet, it becomes totally empty. (Between 0 and 8 feet there is always some water in the bucket.) Compute how much work in foot-pounds is done lifting the bucket.

Answer The bucket empties completely in 8 feet. Therefore at x feet *from the top* the bucket has $10\big(\frac{x}{8}\big)$ pounds of water in it (10 of the initial 11 pounds is water). The weight of bucket+water is $10\big(\frac{x}{8}\big)+1$ pounds. If this is lifted dx feet, the work done is approximately $\big(10\big(\frac{x}{8}\big)+1\big)\,dx$ foot-pounds. The total work is the sum, which is $\int_0^8\big(10\big(\frac{x}{8}\big)+1\big)\,dx = \frac{10}{8}\left(\frac{x^2}{2}\right)+x\Big|_0^8 = \frac{10}{8}\left(\frac{8^2}{2}\right)+8 = 48$. (The last step is not necessary!)

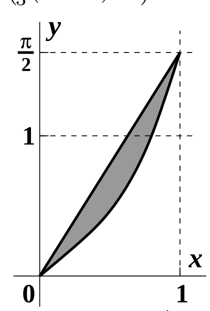
Alternate answer If we measure x from the bucket's initial position, then at height x, the bucket and water contained weighs $11-\frac{10}{8}x$ pounds. We need to lift that weight dx, and then add up. So the total work with this approach is $\int_0^8\big(11-\frac{10}{8}x\big)\,dx = 11x-\frac{10}{8}\left(\frac{x^2}{2}\right)\Big|_0^8 = 88-\frac{10}{8}\left(\frac{8^2}{2}\right)$, and this also is 48.

(10) 4. Find $\int\frac{x}{\sqrt{1-x}}\,dx$.

Answer Suppose $w=1-x$. Then $dw=-dx$ so $-dw=dx$ and $x=1-w$. The integral changes: $\int\frac{x}{\sqrt{1-x}}\,dx = \int\frac{(w-1)}{\sqrt{w}}\,dw = \int w^{1/2}-w^{-1/2}dw = \frac{2}{3}w^{3/2}-2w^{1/2}+C = \frac{2}{3}(1-x)^{3/2}-2(1-x)^{1/2}+C$.

Alternate answer Use integration by parts. Take $u=x$ so $du=dx$ and $dv=\frac{1}{\sqrt{1-x}}\,dx=(1-x)^{-1/2}dx$ so $v=-2(1-x)^{1/2}$. Therefore $\int\frac{x}{\sqrt{1-x}}\,dx = 2x(1-x)^{1/2}+2\int(2-x)^{1/2}dx = 2x(1-x)^{1/2}+2\left(\frac{2}{3}(2-x)^{3/2}\right)+C$.

(14) 5. The graphs of $y=\arcsin(x)$ and $y=\frac{\pi}{2}x$ intersect twice when x is in the interval $[0,1]$. These graphs form the boundary of a region R.



a) Sketch the region R on the axes to the right.

b) Compute the exact value of the area of R. Be sure to evaluate any trig or inverse trig functions which occur in your answer.

Answer with dx: $\int_0^1\big(\frac{\pi}{2}x-\arcsin(x)\big)\,dx$. Use integration by parts to get $\int\arcsin(x)\,dx$. If $u=\arcsin(x)$ and $dv=dx$, then $du=\frac{1}{\sqrt{1-x^2}}\,dx$ and $v=x$. Then $\int\arcsin(x)\,dx = x\arcsin(x)-\int\frac{x}{\sqrt{1-x^2}}\,dx$. Substitute ($w=1-x^2$, $dw=-2x\,dx$ etc.) to get $\int\arcsin(x)\,dx = x\arcsin(x)+\sqrt{1-x^2}+C$. Evaluation: $\int_0^1\big(\frac{\pi}{2}x-\arcsin(x)\big)\,dx = \frac{\pi}{2}\left(\frac{x^2}{2}\right)-\big(x\arcsin(x)+\sqrt{1-x^2}\big)\Big|_0^1 = \big(\frac{\pi}{4}-\arcsin(1)\big)+1 = \big(\frac{\pi}{4}-\frac{\pi}{2}\big)+1 = 1-\frac{1}{4}\pi$.

Also see over for another solution.

Answer with dy (which is certainly a simpler computation!): $y = \arcsin(x)$ becomes $x = \sin y$ and $y = \frac{\pi}{2}x$ becomes $x = \frac{2}{\pi}y$. The area is $\int_0^{\pi/2} \left(\sin(y) - \frac{2}{\pi}y\right) dy = -\cos(y) - \frac{2}{\pi}\left(\frac{y^2}{2}\right)\Big|_0^{\pi/2} = \left(0 - \frac{\pi}{4}\right) - (-1-0) = 1 - \frac{1}{4}\pi$.

(12) 6. Compute $\int_0^\infty (5x+7)\,e^{-x}dx$.

Answer We integrate by parts to get $\int (5x+7)\,e^{-x}dx$. Take $u = 5x+7$ and $dv = e^{-x}dx$, then $du = 5dx$ and $v = -e^{-x}$. So $\int (5x+7)\,e^{-x}dx = (5x+7)(-e^{-x}) + \int e^{-x}5dx = -(5x+7)e^{-x} - 5e^{-x} + C = -(5x+12)e^{-x} + C$. Now $\int_0^\infty (5x+7)\,e^{-x}dx = \lim_{A\to\infty} \int_0^A (5x+7)\,e^{-x}dx$. And $\int_0^A (5x+7)\,e^{-x}dx = -(5x+12)e^{-x}\Big|_0^A = -(5A+12)e^{-A} + 12e^0$. Notice that $\lim_{A\to\infty}(5A+12)e^{-A} = \lim_{A\to\infty}\frac{5A+12}{e^A} \overset{\text{L'H}}{=} \lim_{A\to\infty}\frac{5}{e^A} = 0$ where L'Hôpital's rule is used since both $5A+12$ and $e^A \to \infty$ as $A \to \infty$. The value of the integral is therefore $12e^0 = 12$.

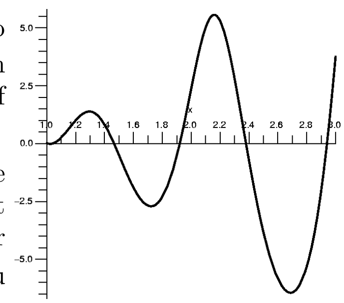
(14) 7. The parts of this problem are not related.

a) Write the Simpson's Rule estimate for $\int_0^6 \sqrt{x^3+x}\,dx$ with $n = 6$ subintervals.

Answer $\left(\frac{6-0}{6\cdot 3}\right)\left(1\sqrt{0^3+0} + 4\sqrt{1^3+1} + 2\sqrt{2^3+2} + 4\sqrt{3^3+3} + 2\sqrt{4^3+4} + 4\sqrt{5^3+5} + 1\sqrt{6^3+6}\right)$

Note The cover page does state **Otherwise do NOT "simplify" your numerical answers!**

b) The function f here is unknown but the Trapezoid Rule approximation to $\int_1^3 f(x)\,dx$ is 2.25537 with $N = 10$ subintervals. A correct graph of f'' is shown to the right. Use the graph and the approximation to find an interval $[A, B]$ of numbers in which the true value of the definite integral must be found.



Graph of $f''(x)$ on $[1,3]$

Answer We know that the Trapezoid Rule error is bounded by $\frac{K_2(b-a)^3}{12N^2}$ where here $a = 1$ and $b = 3$ and $N = 10$. Inspection of the graph of f'' shows that one overestimate of $|f''|$ on $[1,3]$ is about 6, so I'll use 6 for K_2. The error is then bounded by $\frac{6(2^3)}{12(10)^2}$. This number can be used in what follows, or you can "simplify" it to get .04. Then the true value of the integral must be in the interval $[2.25537 - .04, 2.25537 + .04]$.

Note The function which was used to create this problem is $f(x) = \sqrt{1 + (\sin(3x)\ln(x))^2}$ (really!) and further technology shows that a good estimate for the true value of the integral is 2.25948. Many intervals <u>supported by correct reasoning</u> are acceptable here.

For experimental scientists and engineers: you can think of the Trapezoid Rule as giving an *estimate* of some quantity you want to measure. This question asks, essentially, for "error bars" around the estimation: in what interval must the true value actually be? Unlike some of our exam problems (!), in real situations, you aren't given the true value and after that asked to find errors for some approximation.

(12) 8. Verify that $\int_0^1 \frac{1}{\sqrt{16x^2+9}}\,dx = \frac{1}{4}\ln(3)$.

Answer We need $\int \frac{1}{\sqrt{16x^2+9}}\,dx = \frac{1}{4}\int \frac{1}{\sqrt{x^2+\frac{9}{16}}}\,dx$. Take $x = \frac{3}{4}\tan\theta$. Then $dx = \frac{3}{4}(\sec\theta)^2\,d\theta$ and $x^2 + \frac{9}{16} = \frac{9}{16}(\tan\theta)^2 + \frac{9}{16} = \left(\frac{9}{16}\right)\left((\tan\theta)^2+1\right) = \left(\frac{9}{16}\right)(\sec\theta)^2$ and $\frac{1}{4}\int \frac{1}{\sqrt{x^2+\frac{9}{16}}}\,dx$ becomes $\frac{1}{4}\int \left(\frac{1}{\left(\frac{3}{4}\right)\sec\theta}\right)\frac{3}{4}(\sec\theta)^2\,d\theta = \frac{1}{4}\int \sec\theta\,d\theta = \frac{1}{4}\ln(\sec\theta + \tan\theta) + C$. Since $x = \frac{3}{4}\tan\theta$, then $\tan\theta = \frac{4}{3}x$. Because $x^2 + \frac{9}{16} = \left(\frac{9}{16}\right)(\sec\theta)^2$, we see that $\sec\theta = \frac{4}{3}\sqrt{x^2+\frac{9}{16}}$. Therefore $\int \frac{1}{\sqrt{16x^2+9}}\,dx = \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{x^2+\frac{9}{16}} + \frac{4}{3}x\right) + C$. And, finally, evaluation: $\int_0^1 \frac{1}{\sqrt{16x^2+9}}\,dx = \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{x^2+\frac{9}{16}} + \frac{4}{3}x\right)\Big|_0^1 = \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{1^2+\frac{9}{16}} + \frac{4}{3}\cdot 1\right) - \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{0^2+\frac{9}{16}} + \frac{4}{3}\cdot 0\right) = \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{\frac{16}{16}+\frac{9}{16}} + \frac{4}{3}\right) - \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{\frac{9}{16}}\right) = \frac{1}{4}\ln\left(\frac{4}{3}\sqrt{\frac{25}{16}} + \frac{4}{3}\right) - \frac{1}{4}\ln\left(\frac{4}{3}\cdot\frac{3}{4}\right) = \frac{1}{4}\ln\left(\frac{5}{3}+\frac{4}{3}\right) - \frac{1}{4}\ln(1) = \frac{1}{4}\ln\left(\frac{9}{3}\right) - 0 = \frac{1}{4}\ln(3)$.