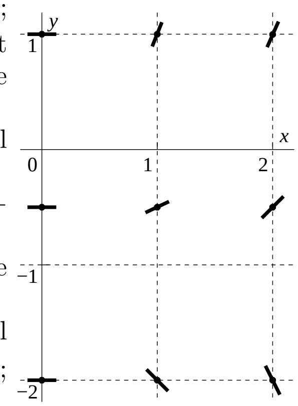
640:152:1, 2, & 3 **Answers to version A of the Second Exam** **4/18/2008**

Here are answers that would earn full credit. Other methods may also be valid.

(16) 1. a) Sketch the elements of the slope field or direction field for the differential equation $\frac{dy}{dx} = x(1+y)$ at the nine points indicated, which are when $x = 0$ and $x = 1$ and $x = 2$ and when $y = -2$ and $y = -\frac{1}{2}$ and $y = 1$.

Answer $\frac{dy}{dx}$ at $(0,-2)$ is 0; $\frac{dy}{dx}$ at $(0,-\frac{1}{2})$ is 0; $\frac{dy}{dx}$ at $(0,1)$ is 0; $\frac{dy}{dx}$ at $(1,-2)$ is -1; $\frac{dy}{dx}$ at $(1,-\frac{1}{2})$ is $\frac{1}{2}$; $\frac{dy}{dx}$ at $(1,1)$ is 2. $\frac{dy}{dx}$ at $(2,-2)$ is -2; $\frac{dy}{dx}$ at $(2,-\frac{1}{2})$ is 1. $\frac{dy}{dx}$ at $(2,1)$ is 4. Line segments of the appropriate slopes are shown on the graph to the right.



b) Find any equilibrium solutions (where y doesn't change) for this differential equation. If there are no such solutions, explain briefly why this is true.

Answer If $y = \texttt{Constant}$ is a solution, then $\frac{dy}{dx} = 0$ for all x so that $0 = x(1 + \texttt{Constant})$ for all x. Yes, there is an equilibrium solution, and it is $y = -1$.

c) Find the solution of the initial value problem $\frac{dy}{dx} = x(1+y)$ and $y(1) = 2$. In the answer express y explicitly as a function of x.

Answer Separate: $\frac{dy}{1+y} = x\,dx$; integrate: $\ln(1+y) = \frac{1}{2}x^2 + C$; use the initial condition: $\ln(1+2) = \frac{1}{2}(1^2) + C$ so $C = \ln(3) - \frac{1}{2}$ and $\ln(1+y) = \frac{1}{2}x^2 + \ln(3) - \frac{1}{2}$; solve for y: $1 + y = e^{\frac{1}{2}x^2+\ln(3)-\frac{1}{2}}$ so $y = e^{\frac{1}{2}x^2+\ln(3)-\frac{1}{2}} - 1$. (Why simplify?)

(12) 2. Calculate the arc length over the given interval: $y = \ln(\cos(x))$, $\left[0, \frac{\pi}{3}\right]$.

Answer If $y = \ln(\cos(x))$, the desired arc length is $\int_0^{\frac{\pi}{3}} \sqrt{1 + (y')^2}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{1}{\cos(x)}(-\sin(x))\right)^2}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{1 + (\tan(x))^2}\,dx = \int_0^{\frac{\pi}{3}} \sqrt{(\sec(x))^2}\,dx = \int_0^{\frac{\pi}{3}} \sec(x)\,dx = \ln(\sec(x) + \tan(x))]_0^{\frac{\pi}{3}} = \ln(\sec(\frac{\pi}{3}) + \tan(\frac{\pi}{3})) - \ln(\sec(0) + \tan(0)) = \ln\left(2 + \sqrt{3}\right) - \ln(1) = \ln\left(2 + \sqrt{3}\right)$.

(12) 3. In this problem $f(x) = x^{3/2}$.

a) Find the second degree Taylor polynomial $T_2(x)$ for $f(x)$ centered at $x = 4$.

Answer If $f(x) = x^{3/2}$ then $f'(x) = \frac{3}{2}x^{1/2}$, $f''(x) = \frac{3}{4}x^{-1/2}$, and $f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$. $T_2(x)$ centered at $x = 4$ is $f(4) + f'(4)(x-4)^1 + \frac{f''(4)}{2}(x-4)^2 = 8 + 3(x-4)^1 + \frac{3}{16}(x-4)^2$.

b) What is $T_2(5)$? **Answer** $8 + 3(5-4)^1 + \frac{3}{16}(5-4)^2$, a fine answer. If you wish, this is $\frac{179}{16}$.

c) Use the Error Bound for Taylor polynomials to estimate the difference between $T_2(5)$ and $5^{3/2}$.

Answer Since $f^{(3)}(x) = -\frac{3}{8}x^{-3/2}$, we know $|f^{(3)}(x)| = \frac{3}{8}x^{-3/2}$, a *decreasing* function of x. On $[4,5]$, the largest value is at $x = 4$, which gives $K = \frac{3}{64}$. The error is then at most $K\frac{(5-4)^3}{3!} = \frac{1}{128}$.

(10) 4. Find the 6$^{\text{th}}$ degree Taylor polynomial for $f(x) = (1+3x)e^{(-x^2)}$ centered at $x = 0$. (This is also called the Maclaurin polynomial.) Your answer should be <u>a polynomial of degree 6</u>.

Answer We know the Taylor polynomials for e^x centered at $x = 0$, and here is one of them: $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$. Now just plug in $-x^2$ for x. The result is $1 + (-x^2) + \frac{1}{2}(-x^2)^2 + \frac{1}{6}(-x^2)^3$ or $1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$. Multiply by $1 + 3x$ and get $1 + 3x - x^2 - 3x^3 + \frac{1}{2}x^4 + \frac{3}{2}x^5 - \frac{1}{6}x^6 - \frac{3}{6}x^7$. We only need the terms up to degree 6, so the answer is $1 + 3x - x^2 - 3x^3 + \frac{1}{2}x^4 + \frac{3}{2}x^5 - \frac{1}{6}x^6$.

(12) 5. a) Does the sequence $\left\{n\left(5^{2/n} - 1\right)\right\}$ converge? If it does, find its limit.

Answer If $a_n = \left\{n\left(5^{2/n} - 1\right)\right\}$, then rewrite: $a_n = \frac{\left(5^{2/n}-1\right)}{\frac{1}{n}} = \frac{e^{\ln(5)(2/n)}-1}{\frac{1}{n}}$. As $n \to \infty$, this becomes $\frac{0}{0}$ so we may try L'Hôpital's rule: $\lim_{n\to\infty} \frac{e^{\ln(5)(2/n)}-1}{\frac{1}{n}} \stackrel{\text{L'H}}{=} \lim_{n\to\infty} \frac{e^{\ln(5)(2/n)}\ln(5)\left(-\frac{2}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} e^{\ln(5)(2/n)}2\ln(5) = (e^0)2\ln(5) = 2\ln(5)$.

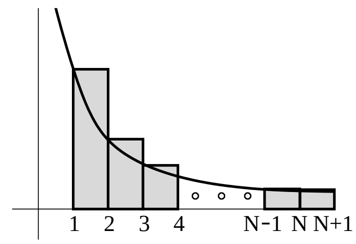
b) Does the series $\sum_{n=1}^{\infty} \frac{4^n + (-3)^n}{5^n}$ converge? If it does, find its sum.

Answer This is a sum of two geometric series. One is $\sum_{n=1}^{\infty} \frac{4^n}{5^n}$, so $c = \frac{4}{5}$ and $r = \frac{4}{5}$: its sum is $\frac{\frac{4}{5}}{1-\frac{4}{5}}$. The other is $\sum_{n=1}^{\infty} \frac{(-3)^n}{5^n}$, so $c = -\frac{3}{5}$ and $r = -\frac{3}{5}$: its sum is $\frac{-\frac{3}{5}}{1+\frac{3}{5}}$. The sum of both series is $\frac{\frac{4}{5}}{1-\frac{4}{5}} + \frac{-\frac{3}{5}}{1+\frac{3}{5}}$, a fine answer. If you wish, this is $\frac{29}{8}$.

(12) 6. The series $\sum_{n=1}^{\infty} \frac{5}{6n+3^n}$ converges and its sum, to an accuracy of .001, is .981. Find a positive integer N so that the partial sum, $S_N = \sum_{n=1}^{N} \frac{5}{6n+3^n}$, has a value within .001 of the sum of the whole series. Explain your reasoning.

Answer The key observation is that $\frac{5}{6n+3^n} < \frac{5}{3^n}$ which is certainly true when n is a positive integer. Then $0 \leq \sum_{n=1}^{\infty} \frac{5}{6n+3^n} - \sum_{n=1}^{N} \frac{5}{6n+3^n} = \sum_{n=N+1}^{\infty} \frac{5}{6n+3^n} < \sum_{n=N+1}^{\infty} \frac{5}{3^n} \overset{\text{geom.}}{\underset{\text{series}}{=}} \frac{\frac{5}{3^{N+1}}}{1-\frac{1}{3}} = \frac{\frac{5}{2}}{3^N}$. Since $.001 = \frac{1}{1{,}000}$ we can consult the table given and see that $N = 8$ will be enough since $\frac{\frac{5}{2}}{6{,}561} < \frac{1}{1{,}000}$.

(12) 7. The infinite series $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$ diverges. Find N so that the partial sum, $\sum_{n=1}^{N} \frac{5}{\sqrt{n}}$, is larger than 100.



Answer Compare the partial sum to an integral. The picture to the right (which is correct since $x^{-1/2}$ is decreasing for $x > 0$) verifies the inequality $\sum_{n=1}^{N} \frac{5}{n^{1/2}} > \int_1^{N+1} \frac{5}{x^{1/2}}\,dx$. Now integrate: $\int_1^{N+1} \frac{5}{x^{1/2}}\,dx = 5\left(2x^{1/2}\right)\Big]_1^{N+1} = 10(N+1)^{1/2} - 10$. To make this *larger than* 100, I'll take $N = 10^6 - 1 = 1{,}000{,}000 - 1 = 999{,}999$ (many other choices work!). Then $N+1 = 10^6$ so that $(N+1)^{1/2} = (10^6)^{1/2} = 10^3$. Surely $10 \cdot 1{,}000 - 10$ is larger than 100.

(14) 8. This problem is about the power series $\sum_{n=1}^{\infty} \left(\frac{2^n}{\sqrt{n}}\right) x^n$.

a) What is the radius of convergence of this power series?

Answer $a_n = \left(\frac{2^n}{\sqrt{x}}\right) x^n$ so $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{2^{n+1}}{\sqrt{n+1}}\right) x^{n+1}}{\left(\frac{2^n}{\sqrt{n}}\right) x^n}\right| = \frac{2^{n+1}\sqrt{n}}{2^n\sqrt{n+1}}|x| = \left(\sqrt{\frac{n}{n+1}}\right)(2|x|)$. As $n \to \infty$, the ratio $\frac{n}{n+1} \to 1$. You can see this using L'Hôpital's rule or just dividing the top and bottom of the fraction by n. So the limit of the ratios is $2|x|$. The radius of convergence therefore must be $\frac{1}{2}$ using the Ratio Test. The Root Test can also be used.

b) What is the behavior of the power series (divergence, absolute or conditional convergence) on the boundary points of the interval of convergence?

Answer We need to check $x = \pm\frac{1}{2}$. If $x = \frac{1}{2}$ then the series becomes $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a p-series with $p = \frac{1}{2} < 1$, so the series diverges.

If $x = -\frac{1}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This satisfies all of the conditions of the Alternating Series Test (the signs alternate, the terms without signs decrease, and the limit of the terms is 0). This series converges, and since the series without signs diverges as we saw before, this series converges conditionally.