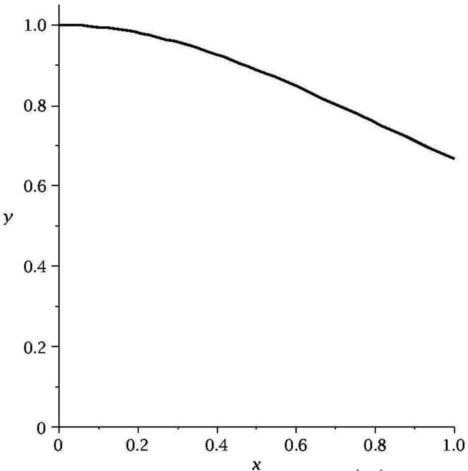
Math 152:1, 2, & 3 April 17, 2008

A problem about the ERROR BOUND for Taylor polynomials

Problem Here $f(x)$ is defined by the formula $f(x) = \frac{2}{x^2+2}$ for x's in the interval $[0,1]$. A graph of $y = f(x)$ is shown to the right.



Graph of $y = f(x)$

a) What are $f(0)$ and $f'(0)$ and $f''(0)$?

Answer Certainly $f(0) = \frac{2}{0^2+2} = 1$. And almost easily, $f'(x) = \frac{-4x}{(x^2+2)^2}$ so that $f'(0) = 0$. Less easily, $f''(x) = \frac{(-4)(x^2+2)^2 - 2(x^2+2)(2x)(-4x)}{(x^2+2)^4}$ and $f''(0) = \frac{(-4)2^2}{2^4} = -1$. A look at the picture should confirm the first two values (height 1 at 0 and horizontal tangent there) and the sign of the third (the curve is concave *down* near 0).

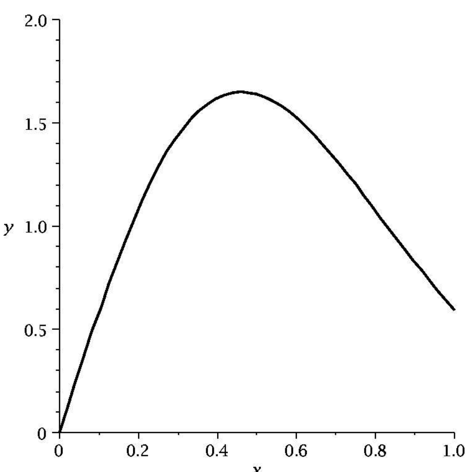
b) What is the second degree Taylor polynomial, $T_2(x)$, centered at $x = 0$ (the MacLaurin polynomial)?

Answer $T_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 = 1 - \frac{1}{2}x^2$.

c) What is $T_2(1)$?

Answer $T_2(1)$ is exactly $1 - \frac{1}{2}(1^2) = \frac{1}{2} = .5$.

d) To the right is a graph of the <u>third derivative of f</u>. Use this graph of $y = f^{(3)}(x)$ to estimate the difference between $T_2(1)$ and $f(1)$. and $f(.5)$.



Graph of $y = f^{(3)}(x)$,
the third derivative of $f(x)$

Answer The **Error Bound** states that $|T_2(1) - f(1)|$ is overestimated by $\frac{K}{3!}|1-0|^3$. This is $\frac{K}{6}$. What is K? It is an overestimate of the absolute value of the third derivative between 1 and 0. This problem was created to show that just looking at end points is *not enough* and actually can lead to computational error. Notice that the true value of $f(1)$ is $\frac{2}{1^2+2} = \frac{2}{3} \approx .66667$. The absolute value of the difference between $f(1)$ and $T_2(1)$ is .16667. Let's see what happens in several "scenarios".

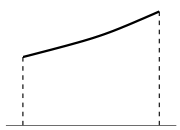
Casual look The left and right endpoint heights of the $f^{(3)}(x)$ graph seem to be at 0 on the left and at $\approx .7$ on the right. So take $K = .7$ and the **Error Bound** seems to be $\frac{.7}{6}$ which is about .11667, and that's certainly smaller than the true value of the difference. This value of K is incorrect!

Sincere look We must consider all of the graph, and try to find the largest value of $|f^{(3)}(x)|$ on all of $[0,1]$. That largest value seems to occur *inside* the interval, somewhere between .4 and .5, and the value of $|f^{(3)}(x)|$ is about 1.7. So K should really be about 1.7, and the true **Error Bound** is about $\frac{1.7}{6} \approx .28333$. This is valid, since $.28333 > .16667$.

OVER

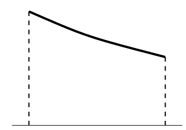
Comments This is *not* too being too fussy over some silly small numbers. If real computation is done to analyze real problems, the answers should come with some confidence that they are correct. If we can't get correct answers in this toy problem, then we'd better think more.
Students may believe that all functions which occur in Taylor error estimates are increasing or decreasing, so consideration of values at the end points is enough to estimate the size over the whole interval. That is certainly not true, and if such logic is used, then students must give some reason to believe their statements.

"It goes up." (An *increasing* function)



Typically this is true for e^x or 2^x (b^x when $b > 1$) or *positive* powers of x like $x^{8/7}$ and x^3 when $x > 0$. Also true for $\ln x$. You can check these statements, if you wish, by considering the first derivative.

"It goes down." (A *decreasing* function)



Typically this is true for e^{-x} or $(\frac{1}{3})^x$ (b^x when $0 < b < 1$) or *negative* powers of x like $x^{-8/7}$ and $\frac{1}{x^3}$ when $x > 0$. You can check these statements, if you wish, by considering the first derivative.

Exam Behavior I'd like to see some acknowledgment by the student that thought has been given to the possibility of bumps "in the middle" as occurred in this example. So a student should write "this function is {in|de}creasing" (whichever is appropriate and correct), maybe accompanied by a simple picture, and then choose the correct end point to get a suitable K. A detailed analysis of the function is *not* required!

More details If $f(x) = \frac{2}{x^2+2}$ then $f^{(3)}(x) = -\frac{48x(x^2-2)}{(x^2+2)^4}$. The maximum value of the third derivative occurs at $x = \frac{\sqrt{50-20\sqrt{5}}}{5} \approx .45951$ and the corresponding value of $f^{(3)}(x)$ is $\frac{75\sqrt{250-100\sqrt{5}}}{4(5-\sqrt{5})^4} \approx 1.65058$. These numbers were obtained by looking at the derivative of the third derivative*, of course. It is amazing (sad?) what can be done with a machine.

* The fourth derivative is $\frac{48(5x^4-20x^2+4)}{(x^2+2)^5}$.