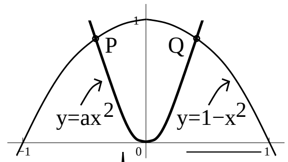
(16) 1. a) Compute the area enclosed by $y = 1 - x^2$ and the x-axis.

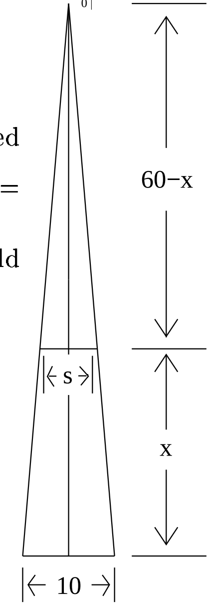
Answer $\int_{-1}^{1} 1 - x^2\,dx = x - \frac{x^3}{3}\Big]_{-1}^{1} = \frac{2}{3} - \left(-\frac{2}{3}\right) = \frac{4}{3}$.



b) Suppose a is an unspecified positive number. Sketch and label the curves $y = 1 - x^2$ and $y = ax^2$ on the axes given. Find the coordinates of the points of intersection of $y = 1 - x^2$ and $y = ax^2$, and label these points on your sketch. **Answer** If $1 - x^2 = ax^2$, then $1 = (1+a)x^2$ so that $x = \pm\frac{1}{\sqrt{1+a}}$. P's coordinates are $\left(-\frac{1}{\sqrt{1+a}}, \frac{a}{1+a}\right)$ and Q's, $\left(\frac{1}{\sqrt{1+a}}, \frac{a}{1+a}\right)$.

c) Find a so that the area enclosed by $y = 1 - x^2$ and $y = ax^2$ is half of the area computed in part a). **Answer** The area between $y = 1 - x^2$ and $y = ax^2$ is $\int_{-\frac{1}{\sqrt{1+a}}}^{\frac{1}{\sqrt{1+a}}} (1 - x^2) - ax^2\,dx = \int_{-\frac{1}{\sqrt{1+a}}}^{\frac{1}{\sqrt{1+a}}} 1 - (1+a)x^2\,dx = x - (1+a)\frac{x^3}{3}\Big]_{-\frac{1}{\sqrt{1+a}}}^{\frac{1}{\sqrt{1+a}}} = \frac{2}{\sqrt{1+a}} - \frac{2}{3\sqrt{1+a}} = \frac{4}{3\sqrt{1+a}}$. This should be $\frac{2}{3}$. So $\frac{2}{3} = \frac{4}{3\sqrt{1+a}}$. Then $\sqrt{1+a} = 2$ and $1 + a = 4$, so, finally, a must be 3. Wow!

(12) 2. A flat-sided monolith* is 60 feet tall with a square base that is 10 feet on each side. What is the volume of the monolith? **Answer** Our coordinate system's origin is at the center of the base of the monolith. We see a slice through the central axis of this solid. The height ranges from 0 to 60, and the width of the square cross-sections, from 10 to 0. If the height is x and the side of the cross-section is s then $\frac{60-x}{60} = \frac{s}{10}$ so $s = 10 - \frac{1}{6}x$. The volume is the sum of cross-section areas, $A(x) = s^2$, multiplied by a bit of height (dx)so the volume is $\int_0^{60} A(x)\,dx = \int_0^{60} \left(10 - \frac{1}{6}x\right)^2\,dx$. This is $\left(\frac{-1}{-\frac{1}{6}}\right)\frac{1}{3}(10)^3 = 2{,}000$.



(18) 3. a) Here's a formula from the *Tables of Indefinite Integrals* by G. Petit Bois (1906): $\int \frac{x^2}{x^3+5x^2+8x+4}\,dx = \log(x+1) + \frac{4}{x+2}$. Please verify this formula using the method of partial fractions.

Answer The formula *can* be verified by differentiation and algebraic manipulation, but you're asked to decompose the integrand using partial fractions. The formula, which has $x + 1$, is a hint. I *guess* that -1 is a root of $x^3 + 5x^2 + 8x + 4$. If we plug in $x = -1$, the value is $(-1)^3 + 5(-1)^2 + 8(-1) + 4 = 0$. Divide $x^3 + 5x^2 + 8x + 4$ by $x + 1$. The quotient is $x^2 + 4x + 4 = (x+2)^2$. The partial fraction expansion for $\frac{x^2}{x^3+5x^2+8x+4}$ is $\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$ and this is $\frac{A(x+2)^2+Bx(x+2)+C(x+1)}{(x+1)(x+2)^2}$. We need A, B, and C so that $A(x+2)^2 + Bx(x+2) + C(x+1) = x^2$. If $x = -1$, then $A = 1$. If $x = -2$, then $-C = 4$ so $C = -4$. Consider the x^2 coefficient: then $A + B = 1$ and since $A = 1$, B must be 0. Therefore $\frac{x^2}{x^3+5x^2+8x+4} = \frac{1}{x+1} + \frac{-4}{(x+2)^2}$. An antiderivative of $\frac{1}{x+1}$ is $\ln|x+1|$ and an antiderivative of $\frac{-4}{(x+2)^2}$ is $\frac{4}{x+2}$. We have verified the formula.

b) Here's another formula from the same text: $\int \frac{x^2}{x^3+x^2+x+1}\,dx = \frac{1}{2}\log\left((x+1)\sqrt{x^2+1}\right) - \frac{1}{2}\arctan x$. Again, please verify this formula using the method of partial fractions.

Answer Again, the formula gives a clue: let $x = -1$ in $x^3 + x^2 + x + 1$ then $(-1)^3 + (-1)^2 + (-1) + 1 = 0$ so divide $x^3 + x^2 + x + 1$ by $x + 1$: the quotient is $x^2 + 1$. The form of the partial fraction expansion is $\frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+1)(x+1)}{(x+1)(x^2+1)}$. This is $\frac{x^2}{x^3+x^2+x+1}$ when $x^2 = A(x^2+1) + (Bx+C)(x+1)$. If $x = -1$ we see $A = \frac{1}{2}$. The x^2 coefficients give $A + B = 1$ so $B = \frac{1}{2}$. The constant terms give C: $0 = A + C$ so $C = -\frac{1}{2}$. Antidifferentiate $\frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x-\frac{1}{2}}{x^2+1}$: $\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x$. Since $\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln(x^2+1) = \frac{1}{2}\ln\left((x+1)\sqrt{x^2+1}\right)$, we're done. (The formulas in this problem are on page 12 of the Dover reprint.)

(18) 4. a) Compute $\int_0^1 x\arcsin(x^2)\,dx$. **Answer** Integrate by parts, using $\left.\begin{array}{l} u = \arcsin(x^2) \\ dv = x\,dx \end{array}\right\}\left\{\begin{array}{l} du = \frac{1}{\sqrt{1-x^4}}2x\,dx \\ v = \frac{1}{2}x^2 \end{array}\right.$.

Then $\int u\,dv = uv - \int v\,du$ becomes $\int x\arcsin(x^2)\,dx = \frac{1}{2}x^2\arcsin(x^2) - \int \frac{x^3}{\sqrt{1-x^4}}\,dx$. An antiderivative of the last part is obtained using the substitution $w = 1 - x^4$. All this becomes $\int_0^1 x\arcsin(x^2)\,dx = \frac{1}{2}x^2\arcsin(x^2) + \frac{1}{2}\sqrt{1-x^4}\Big]_0^1 = \frac{1}{2}\arcsin(1) - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}$. Alternatively, substitute $w = x^2$, then $\int u\,dv$, etc.

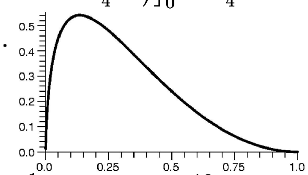
OVER

* **monolith** 1. A large block of stone, especially one used in architecture or sculpture.
2. Something, such as a column or monument, made from one large block of stone.

b) Compute $\int_0^1 \frac{e^{2x}}{e^x+1}\,dx$. **Answer** Use the substitution $w = e^x + 1$ so $dw = e^x\,dx$ and $e^x = w - 1$. Then $\frac{e^{2x}}{e^x+1}\,dx = \frac{e^x}{e^x+1}\,e^x dx = \frac{w-1}{w}\,dw = 1 - \frac{1}{w}\,dw$ which has antiderivative $w - \ln|w|$ so $\int_0^1 \frac{e^{2x}}{e^x+1}\,dx = e^x + 1 - \ln(1+e^x)]_0^1 = (e+1) - \ln(e+1) - (2 - \ln(2))$.

(16) 5. In this problem, $f(x) = x\,(\ln x)^2$. a) Verify that $\lim_{x\to 0+} f(x) = 0$. **Hint** Write the limit so you can apply L'H, but be sure to indicate *why* you need L'H *whenever* you use it. **Answer** $\lim_{x\to 0+} x\,(\ln x)^2 = \lim_{x\to 0+} \frac{(\ln x)^2}{\frac{1}{x}}$. This is $\frac{\infty}{\infty}$, so we may apply L'Hopital's rule. We consider the limit of the derivative of the top over the derivative of the bottom: $\lim_{x\to 0+} \frac{2(\ln x)\left(\frac{1}{x}\right)}{-\frac{1}{x^2}} = \lim_{x\to 0+} \frac{-2\ln x}{\frac{1}{x}}$. Again, $\frac{\infty}{\infty}$, so again L'H and then $\lim_{x\to 0+} \frac{\frac{-2}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0+} 2x = 0$.

b) Carefully compute the improper integral $\int_0^1 f(x)\,dx$. Indicate why the limits you need exist, and what these limits are. **Answer** Integration by parts: $\int u\,dv = uv - \int v\,du$ becomes $\left.\begin{array}{c} u = (\ln x)^2 \\ dv = x\,dx \end{array}\right\}\left\{\begin{array}{c} du = 2(\ln x)\left(\frac{1}{x}\right)dx \\ v = \frac{1}{2}x^2 \end{array}\right.$ so $\int x\,(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$. But $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$ with another integration by parts: $\left.\begin{array}{c} u = \ln x \\ dv = x\,dx \end{array}\right\}\left\{\begin{array}{c} du = \left(\frac{1}{x}\right)dx \\ v = \frac{1}{2}x^2 \end{array}\right.$. Thus $\int_0^1 x\,(\ln x)^2\,dx = \frac{1}{2}x^2(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right)\Big]_0^1 = \frac{1}{4}$ since $\lim_{x\to 0+} x^2(\ln x)^2 = \lim_{x\to 0+} x\cdot x(\ln x)^2 = 0$ as we saw and, similarly, $\lim_{x\to 0+} x^2(\ln x) = 0$.



c) Here's a graph of $x(\ln x)^2$ drawn by `Maple` on the interval $[0,1]$. Does this graph (approximately) confirm your computation in b)? Why? **Answer** Yes. It's almost a triangle of height $\frac{1}{2}$, base 1, and area $\frac{1}{4}$, which is b)'s answer.

(10) 6. The integral $\int_2^\infty \frac{1}{4x^{1/3}+5x^{1/4}}\,dx$ diverges to ∞. Find some $A > 0$ so that $\int_2^A \frac{1}{4x^{1/3}+5x^{1/4}}\,dx > 10^{10}$. **Comment** You are *not* asked to find an explicit antiderivative of $\frac{1}{4x^{1/3}+5x^{1/4}}$**. You are *not* asked to find a "best possible" A. You *are* asked to find a valid A and to support your answer with some reasoning. **Answer** If $x \ge 1$, then $1 \le x^{1/4} \le x^{1/3}$. Therefore $\frac{1}{4x^{1/3}+5x^{1/4}} \ge \frac{1}{4x^{1/3}+5x^{1/3}} = \frac{1}{9x^{1/3}}$ and $\int_2^A \frac{1}{4x^{1/3}+5x^{1/4}}\,dx > \int_2^A \frac{1}{9x^{1/3}}\,dx = \frac{1}{9}\cdot\frac{3}{2}x^{2/3}\Big]_2^A = \frac{1}{6}\left(A^{3/4} - 2^{3/4}\right)$. I'd like $\frac{1}{6}\left(A^{2/3} - 2^{2/3}\right) > 10^{10}$. Just unroll, and don't *worry* about pretty numbers: one A to choose is $(2^{2/3} + 6\cdot 10^{10})^{3/2} + 1$.

(10) 7. This problem analyzes the computation needed to estimate the definite integral $\int_0^2 x\left(1+x^3\right)^{3/2}dx$ using the Trapezoidal Rule. Find n (the number of subdivisions) so that the Trapezoidal Rule estimate will be within 10^{-6} of the true value of the definite integral. (You may use the error bound $\frac{K(b-a)^3}{12n^2}$ where K is an overestimate of the magnitude of the second derivative.) **Comment** You are *not* asked to compute this approximation to the definite integral. You are *not* asked to find a "best possible" n. You *are* asked to find a valid n and to support your answer with some reasoning. **Answer** If $f(x) = x\left(1+x^3\right)^{3/2}$ then $f'(x) = \left(1+x^3\right)^{3/2} + x\left(\frac{3}{2}\right)(1+x^3)^{1/2}(3x^2) = \left(1+x^3\right)^{3/2} + \left(\frac{9}{2}\right)x^3(1+x^3)^{1/2}$ and $f''(x) = \left(\frac{3}{2}\right)(1+x^3)^{1/2}(3x^2) + \left(\frac{27}{2}\right)x^2(1+x^3)^{1/2} + \left(\frac{9}{4}\right)x^3(1+x^3)^{-1/2}(3x^2)$. We need an overestimate of $|f''(x)|$ for x in $[0,2]$. Consider the first two terms of $f''(x)$: $\left(\frac{3}{2}\right)(1+x^3)^{1/2}(3x^2) + \left(\frac{27}{2}\right)x^2(1+x^3)^{1/2}$. All of these formulas have positive exponents and positive coefficients, so all are *increasing* on $[0,2]$. An overestimate is obtained just by plugging in $x = 2$. Amazingly† $(1+2^3)^{1/2} = 3$ and $\left(\frac{3}{2}\right)(1+x^3)^{1/2}(3x^2) + \left(\frac{27}{2}\right)x^2(1+x^3)^{1/2}$ is estimated by $\left(\frac{3}{2}\right)3(3\cdot 2^2) + \left(\frac{27}{2}\right)2^2 3$. We can compute this: it is 216. Now for $\left(\frac{9}{4}\right)x^3(1+x^3)^{-1/2}(3x^2) = (1+x^3)^{-1/2}\left(\frac{27}{4}\right)x^5$. To overestimate this simply, take it apart: $(1+x^3)^{-1/2}$ is decreasing on $[0,2]$ and has maximum value when $x = 0$, and is 1 there. $\left(\frac{27}{4}\right)x^5$ is increasing on $[0,2]$ with maximum value $\left(\frac{27}{4}\right)2^5 = 216$. An overestimate of the last part of $|f''(x)|$ is 216. We can take $K = 432$ in our formula. Since $b = 2$ and $a = 0$, $\frac{K(b-a)^3}{12n^2}$ becomes $\frac{432(2^3)}{12n^2} = \frac{288}{n^2}$. This to be less than 10^{-6} if $\frac{288}{n^2} < 10^{-6}$ or $n > \sqrt{288\cdot 10^6}$. So $n = 17{,}000$ should be enough. **Comment** An approximate value of the integral is 19.32654961. The trapezoid rule with $n = 100$ gives the answer 19.33…, with $n = 1{,}000$ gives 19.32659…, and finally with $n = 17{,}000$ gives 19.3265497…. That computation took a relatively long time (almost 3 seconds). This $f(x)$ is bad (from the trapezoid rule point of view) near $x = 0$. By the way, a graph of f'' with `Maple` suggests that actually $K \approx 289$ so our "lazy" estimate (432) wasn't too bad.

** `Maple` showed an antiderivative with 23 complicated terms. This was not useful.

† Not really. This *is* an exam without calculators.