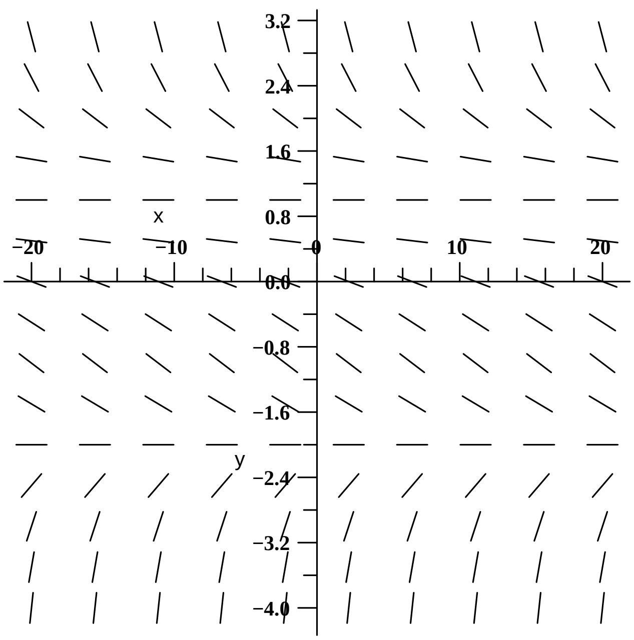
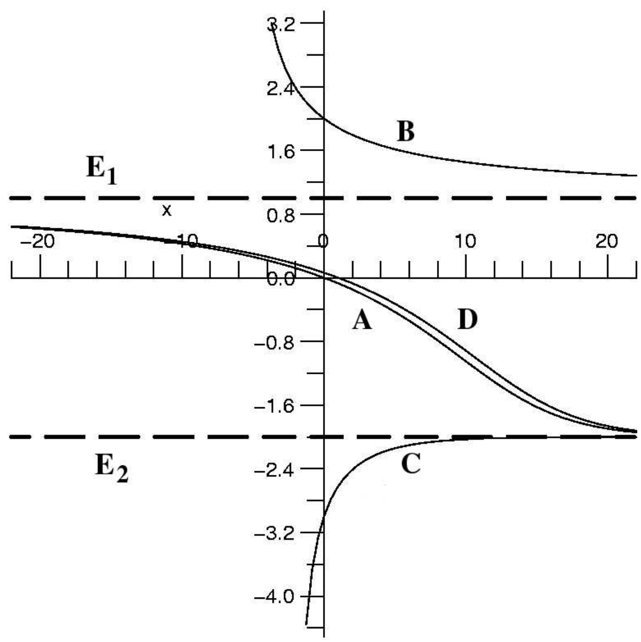
640:192:01 **Answers to the Second Exam** **11/25/2005**

(12) 1. The horizontal and vertical axes on the graph below have different scales. The graph is a direction field for to the autonomous differential equation $y' = -\frac{1}{30}(y-1)^2(2+y)$.
a) Find the equilibrium solutions (where y doesn't change) for this differential equation.
Answer Here $y' = 0$ always so y must be constant. The solutions are $y(x) = 1$ (which is labeled $\mathbf{E_1}$ below) and $y(x) = -2$ (labeled $\mathbf{E_2}$).
b) Sketch solution curves on the axis above through these points. Find the indicated limits:

- $(0,0)$. Label this curve **A**. On curve **A**, $\lim_{x\to\infty} y(x) =$ ___-2___.
- $(0,2)$. Label this curve **B**. On curve **B**, $\lim_{x\to\infty} y(x) =$ ___1___.
- $(0,-3)$. Label this curve **C**. On curve **C**, $\lim_{x\to\infty} y(x) =$ ___-2___.
- $(1,0)$. Label this curve **D**. On curve **D**, $\lim_{x\to\infty} y(x) =$ ___-2___.





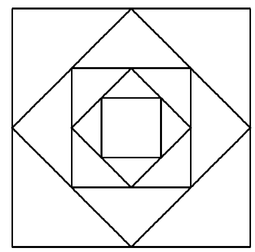
Answer I learned a new `Maple` command (`fieldplot`) to draw the direction field: there are 150 little line segments which I didn't want to draw by hand. Rather than draw the solution curves by hand, I again used `Maple` (with the commands `dsolve`, a sophisticated symbolic and numerical ODE solver, and `odeplot`) to draw the solution curves shown. I had one surprise, which was how close the curves **A** and **D** seemed. In retrospect, the graphs almost touch because of the difference in the horizontal and vertical scales.
c) One of the equilibrium solutions is a *stable* equilibrium. Which one?
Answer $y = -2$, which I called $\mathbf{E_2}$.

(15) 2. Consider the differential equation $y' = \frac{1+y}{x^3}$. a) Find the general solution.
Answer This equation is separable, and that realization leads immediately to $\int \frac{dy}{1+y} = \int \frac{dx}{x^3}$, so that $\ln(1+y) = -\frac{1}{2x^2} + C$. Exponentiating and renaming $e^C = K$ leads to $1 + y = Ke - \frac{1}{2x^2}$ so that $y = Ke^{-\frac{1}{2x^2}} - 1$. Although in this derivation K could not be 0 since it is a value of the exponential function, the curve defined by $y = -1$ (with $K = 0$) is also a solution, the only equilibrium solution of the equation.

b) Find a particular solution with $y(1) = 0$ and describe its domain. **Comment** The domain is slightly tricky. **Answer** Plug $(1,0)$ in $y = Ke^{-\frac{1}{2x^2}} - 1$ to get $0 = Ke^{-\frac{1}{2}} - 1$. So $K = \frac{1}{e^{-\frac{1}{2}}} = e^{\frac{1}{2}}$ and the solution is $y = e^{\frac{1}{2}}e^{-\frac{1}{2x^2}} - 1$. The domain of the function defined by the formula on the right-hand side of the equation is all non-zero x's. But the solution specified by $y(1) = 0$ is uniquely defined by the formula for $x > 0$. Other K's can be used for $x < 0$. Thus the domain of the solution is $(0, +\infty)$, the part of the graph of the function which is *connected* to $(1,0)$. This is similar to several drawing in problem 1.

(15) 3. The parts of this problem are not related. a) The first term of a geometric series is 5 and the fourth term is 3. What is the sum of the geometric series?

Answer A geometric series looks like: $a + ar + ar^2 + ar^3 + \ldots$ that $a = 5$ and $ar^3 = 3$. Therefore $r^3 = \frac{3}{5}$ and $r = \left(\frac{3}{5}\right)^{\frac{1}{3}}$. The sum is $\frac{a}{1-r} = \frac{5}{1-\left(\frac{3}{5}\right)^{\frac{1}{3}}}$.



b) An infinite sequence of squares is drawn (the first five are shown), with the midpoints of the sides of one being the vertices of the next. The outermost square has sides which are 1 unit long. What is the sum of the perimeters of all of the squares?

Answer The perimeter of the outermost square is 4. The length of the next square's side inward is $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}$. The perimeter of that square is $\frac{4}{\sqrt{2}}$. The ratio of successive perimeters is $\frac{1}{\sqrt{2}}$ (clearly?). The sum of the perimeters is a geometric series with first term equal to 4. The sum of the perimeters must be $\frac{4}{1-\frac{1}{\sqrt{2}}}$.

(12) 4. a) What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^n}{n^2}x^n$?

Answer I'll use the Ratio Test (the Root Test works well, also). Then $a_n = \frac{3^n|x|^n}{n^2}$ and $a_{n+1} = \frac{3^{n+1}|x|^{n+1}}{(n+1)^2}$ so (after algebraic manipulation) $\frac{a_{n+1}}{a_n} = 3|x|\frac{n^2}{(n+1)^2}$. Since $\frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2}$, its limit is 1 as $n \to \infty$ (or use l'Hopital). Therefore $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 3|x|$. The series converges absolutely if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. The radius of convergence is $\frac{1}{3}$.

b) What is the behavior of the power series (divergence, absolute or conditional convergence) on the boundary points of the interval of convergence?

Answer We need to look at $x = -\pm\frac{1}{3}$. If $x = \frac{1}{3}$ the series becomes $\sum_{n=1}^{\infty} \frac{1}{n^2}$, a $p-$series with $p = 2 > 1$ which converges. Since the series is positive, it converges absolutely. When $x = -\frac{1}{3}$, we get $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ which converges absolutely and therefore converges.

(12) 5. In this problem you will consider the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln(n))^2}$.

a) Briefly explain why this series converges.

Answer This is an alternating series. We try the Alternating Series Test. The signs of the terms alternate because of the $(-1)^n$. The absolute value of a term is $\frac{1}{n(\ln(n))^2}$ which decreases, and indeed decreases to 0.* The series converges by the Alternating Series Test.

* A very pedantic person could check this by looking at the derivative of $f(x) = \frac{1}{x(\ln(x))^2}$ in the interval $[2, +\infty)$, but I know both x and $\ln x$ are increasing there and both have limits $+\infty$ so the function has limit 0.

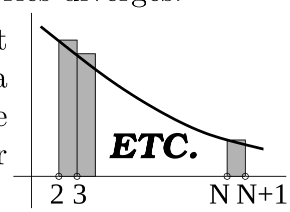
b) `Maple` gives the approximate value .84776 to 5 digit accuracy for the sum of this series. Find a specific partial sum which is guaranteed to give this number to 5 digit accuracy. Give evidence supporting your assertion.

Comment You are *not* asked for the best possible partial sum satisfying the indicated requirement! You must, however, give supporting evidence for the partial sum you give.

Answer A partial sum of a convergent alternating series is accurate to within the absolute value of the first omitted term. So if we want $\frac{1}{(N+1)(\ln(N+1))^2}$ to be less than .000001 (yes, why not 5 zeros, so I don't need to worry about what "5 digit accuracy" means), I could just take $N = 10^6 + 1$. The logarithms are > 1 so they just help. The partial sum $\sum_{n=2}^{10^5+1} \frac{(-1)^n}{n(\ln(n))^2}$ works.

(12) 6. In this problem you will consider the series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$. a) Briefly explain why this series <u>diverges</u>. **Answer** One "natural" approach is the Integral Test. $f(x) = \frac{1}{x\sqrt{\ln x}}$ is a positive decreasing function in $[0, +\infty)$. We compute $\int_2^Q \frac{1}{x\sqrt{\ln x}}dx = 2\sqrt{\ln x}\Big]_2^Q = 2\sqrt{\ln(Q)} - 2\sqrt{\ln 2}$. As $q \to +\infty$, $\sqrt{\ln Q} \to \infty$ also. Therefore by the Integral Test, the series diverges.

b) According to `Maple`, the 10,000$^{\text{th}}$ partial sum of this series is about 4.74561. Are the partial sums of this series unbounded? If yes, find a specific partial sum which is guaranteed to be greater than 100. Give evidence supporting your assertion. **Comment** You are *not* asked for the best possible partial sum satisfying the indicated requirement! You must, however, give supporting evidence for the partial sum you give.



Answer The picture shows (I generally draw the picture, but the formula sheet also has supporting material) that $\int_2^{N+1} \frac{1}{x\sqrt{\ln x}}dx < \sum_{n=2}^{N} \frac{1}{n\sqrt{\ln(n)}}$. We have a partial sum as desired if the integral is at least 100. This is true by what's in a) when $2\sqrt{\ln(N+1)} - 2\sqrt{\ln 2} > 100$ so $\ln(N+1)$ should be at least $50+\ln 2$: make it 51 since $0 < \ln 2 < 1$. Therefore $N+1 > e^{51}$. Use the partial sum $\sum_{n=2}^{N} \frac{1}{n\sqrt{\ln(n)}}$. where N is any integer $> e^{51} - 1$.

(12) 7. In this problem you will consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+2^n}$.

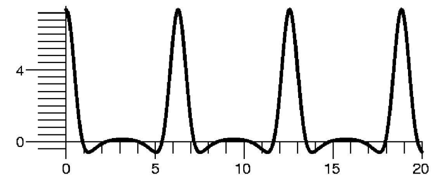
a) Briefly explain why this series <u>converges</u>.

Answer Since $0 \le \frac{1}{\sqrt{n}+2^n} \le \frac{1}{2^n}$ and a geometric series with ratio $\frac{1}{2} < 1$ converges, the series converges.

b) `Maple` gives the approximate value .73601 to 5 digit accuracy for the sum of this series. Find a specific partial sum which is guaranteed to give this number to 5 digit accuracy. Give evidence supporting your assertion. **Comment** You are *not* asked for the best possible partial sum satisfying the indicated requirement! You must, however, give supporting evidence for the partial sum you give.

Answer Consider the tail, $\sum_{n=N+1}^{\infty} \frac{1}{\sqrt{n}+2^n}$. It is less than a geometric series with first term $a = \frac{1}{2^{N+1}}$ and ratio $r = \frac{1}{2}$. Therefore the tail has sum less than $\frac{a}{1-r} = \frac{\frac{1}{2^{N+1}}}{1-\frac{1}{2}} = \frac{1}{2^N}$. I can make this less than $.000001 = 10^{-6}$ by asking that N be at least 24 since $24 = 6 \cdot 4$ and $2^{-4} < \frac{1}{10}$. So the partial sum $\sum_{n=1}^{24} \frac{1}{\sqrt{n}+2^n}$ will suffice.

(12) 8. This problem investigates the function $f(x)$ defined by the sum $f(x) = \sum_{n=0}^{\infty} \frac{2^n \cos(nx)}{n!}$. This is <u>not</u> a power series. Below is a graph of a high partial sum of the series for $0 \le x \le 20$.



a) Does this series converge for all x? If yes, explain why.
Answer We prove that the series converges absolutely and so converges for all x. $|\cos(\text{ANYTHING})| \le 1$ so $\left|\frac{2^n \cos(nx)}{n!}\right| \le \frac{2^n}{n!}$ The series with n^{th} term is $\frac{2^n}{n!}$ converges by the Ratio Test: the ratio between successive terms is $\frac{2}{n+1}$ and this $\to 0 < 1$ as $n \to \infty$.

b) Is the apparent periodicity of the function actually correct? If yes, explain why.

Answer Yes, $f(x)$ is periodic with period 2π. For integer n, $\cos(n(x+2\pi)) = \cos(nx + 2n\pi) = \cos(nx)$ since cosine is 2π periodic. So all the terms in the infinite series for $f(x+2\pi)$ are identical to the terms in the infinite series for $f(x)$.

c) Is the function actually bounded? That is, can you find some positive number B so that $|f(x)| \le B$ for all x? **Comment** You are *not* asked for the best possible bound. If you think a bound exists, find one such bound, and give supporting evidence for your assertion. Otherwise, explain why a bound does not exist.

Answer Yes, the function is bounded. a) shows that one bound is $\sum_{n=0}^{\infty} \frac{2^n}{n!}$. Since this is a convergent series, it converges to *something* which is a bound for $|f(x)|$. **But** this adequate answer, to which I will give full credit, isn't what I wanted. This sum is $e^2 \approx 7.38906$. I know I *wanted* an approximation (an overestimate!) by some simple rational number. I did not ask the question correctly. The response I wanted was something like the following: $|f(x)| \le \sum_{n=0}^{\infty} \frac{2^n}{n!} = \sum_{n=0}^{N} \frac{2^n}{n!} + \sum_{n=N+1}^{\infty} \frac{2^n}{n!}$. The tail can be overestimated by a geometric series if $N+1 \ge 4$: $\sum_{n=4}^{\infty} \frac{2^n}{n!} \le \sum_{j=0}^{\infty} \frac{2^4}{4!} \frac{1}{2^j} = \frac{2^4}{4!} \cdot \frac{1}{1-\frac{1}{2}} = \frac{2^5}{4!}$. This is because for $n \ge 4$, $\frac{2}{n} \le \frac{1}{2}$. A bound for $|f(x)|$ obtained is $\sum_{n=0}^{3} \frac{2^n}{n!} + \frac{2^5}{4!}$, an easily computable rational number. To get that response, I should have asked a different question. The graph is the 100^{th} partial sum. I could have asked if it was within .01 units of the graph $y = f(x)$.

(12) 9. True or false? If false, give an example to show that the implication is not true. If true, briefly explain why. a) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ diverges. **Answer** Since the first sum converges, $\lim_{n\to\infty} a_n = 0$. Then $\frac{1}{a_n}$ is positive and must $\to \infty$ as $n \to \infty$, so the second sum can't converge. The implication is true.

b) Suppose that all of the a_n's are positive. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.
Answer The implication is false. One example is $a_n = 1$ for all n. Then the two sums are equal and both diverge. Another example could be gotten from the harmonic series.