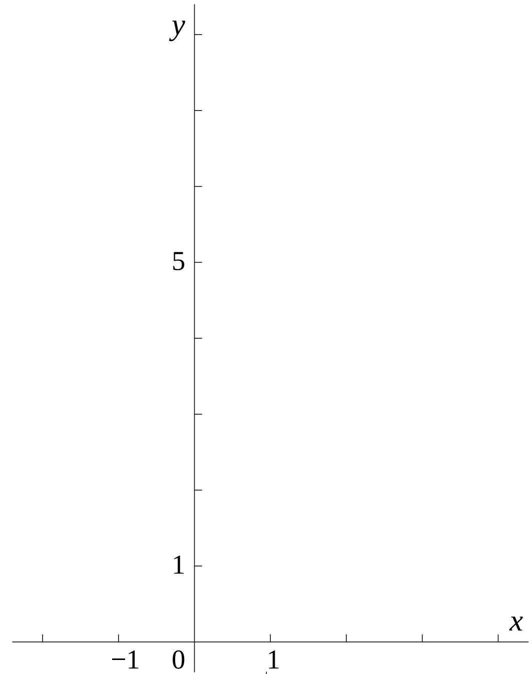
(20) 1. a) Sketch $y = \frac{2}{3}x + 3$ and $y = \frac{1}{3}x^2 + \frac{1}{3}$ on the axes to the right.

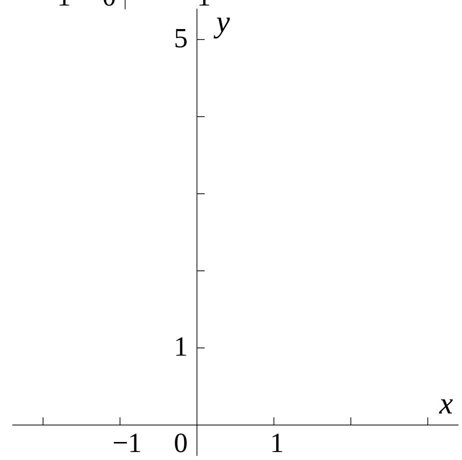
b) Find the points where these curves intersect. Compute the area between the curves.



c) The motion of two points, $\mathcal{A}$ and $\mathcal{B}$, is described by these parametric equations:

$$\mathcal{A}: \begin{cases} x = \boldsymbol{t^3} \\ y = \frac{2}{3}\boldsymbol{t^3} + 3 \end{cases} \qquad \mathcal{B}: \begin{cases} x = \mathbf{cos}\,\boldsymbol{t} \\ y = \frac{1}{3}\left(\mathbf{cos}\,\boldsymbol{t}\right)^2 + \frac{1}{3} \end{cases}$$

Sketch the paths of these points as well as possible on the axes to the right. Label the paths with $\mathcal{A}$ and $\mathcal{B}$.



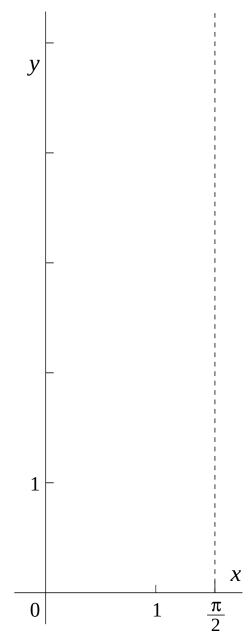
d) Do the points ever collide? Explain your answer briefly.

(16) 2A silicon friend declares that $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx = \ln(2)$. Check that this is correct.

(16) 3. Suppose $\mathcal{S}$ is the three-sided region in the first quadrant bounded by the $y-$axis and the two curves $y = \tan x$ and $y = \sec x$.

a) Sketch that part of the region between $y = 0$ and $y = 5$ on the axes given.

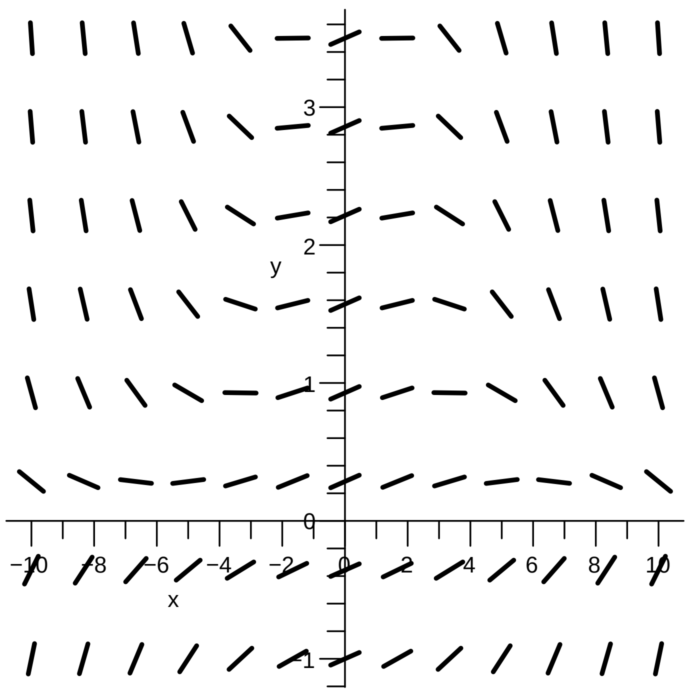
b) Compute the area of the whole region $\mathcal{S}$ (up to $+\infty$ in y) if it is finite.



(16) 4. a) Suppose m and n are positive integers. Find a reduction formula for $\int x^m (\ln x)^n \, dx$.
Comment Here the object is to reduce n, since if we can push n to 0 we'll just have a polynomial to integrate, which is easy.
b) Use the formula obtained in a) to compute $\int x^{20} (\ln x)^2 \, dx$.

(16) 5. a) Find $\displaystyle\int \frac{e^{2x}}{\sqrt{e^{2x}+1}} dx$.
b) Find $\displaystyle\int \frac{e^{x}}{\sqrt{e^{2x}+1}} dx$.
Comment These antiderivatives may appear similar, but different methods are needed.

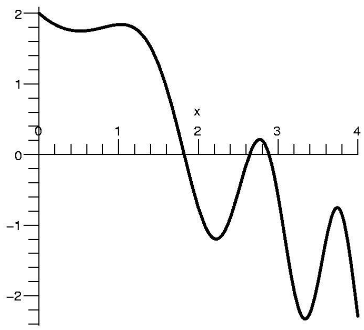
(16) 6. The horizontal and vertical axes on the graph below have different scales. x goes from -10 to 10 and y goes from -1 to 3.5. The graph is a direction field for the differential equation $y' = \frac{1}{10}\left(1 - \frac{1}{10} y x^2\right)$.



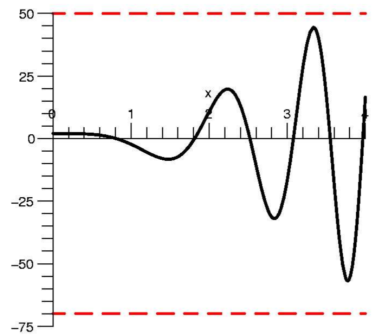
a) Sketch the solution curve which passes through $(0, 1)$ **on the graph above**.
b) How many critical points does this solution curve seem to have? What types of critical points do they seem to be? If (x_0, y_0) is a critical point, find an exact algebraic relationship between x_0 and y_0.
Comment The equation *cannot* be solved in terms of "nice" functions. Information from the graph and the differential equation should be used.

(16) 7. Find a solution of $y' = \dfrac{y}{1-x^2}$ which passes through $(0, 1)$. Write the solution explicitly as $y = \text{FUNCTION}(x)$. What is the domain of the function describing the solution curve?

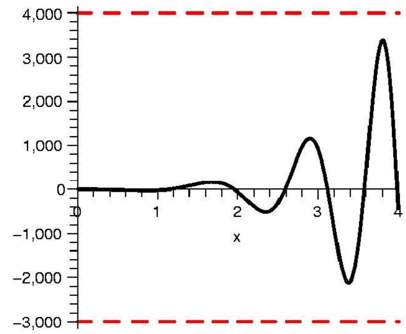
(16) 8. In this problem, $f(x) = 2 - x + \sin(x^2)$. Assume that the graphs of the functions below on the interval $[0, 4]$ are correct. Information from these graphs may be used to answer the questions which follow.



The graph of $y = f(x)$



The graph of $y = f''(x)$, the second derivative of f



The graph of $y = f^{(4)}(x)$, the fourth derivative of f

a) How many subdivisions are needed to estimate $\int_0^4 f(x)\,dx$ with the Trapezoid Rule to an accuracy of 10^{-10}? **Suggestion** Use the picture and the formula sheet.

b) How many subdivisions are needed to estimate $\int_0^4 f(x)\,dx$ with Simpson's Rule to an accuracy of 10^{-10}? **Suggestion** Use the picture and the formula sheet.

(20) 9. Stay calm in this problem, and just do each piece as it comes.

Important The result of any part can be used to prove those parts which *follow* it.

a) Prove: if $1 < A$, then $-1 < -\frac{1}{A} < 0$.

b) Prove: if $1 < A < 100$, then $1 < 2A - \frac{1}{A} < 200$.

c) Prove: if $1 < A < 100$, then $1 < \sqrt{2A - \frac{1}{A}} < 100$.

d) Prove: if $1 < A < B$, then $-1 < -\frac{1}{A} < -\frac{1}{B} < 0$.

e) Prove: if $1 < A < B$, then $1 < \sqrt{2A - \frac{1}{A}} < \sqrt{2B - \frac{1}{B}}$.

f) State an axiom about the real numbers. The axiom should contain the following words in a complete English sentence (the words are in alphabetical order here, and in a different order in the axiom):

... ABOVE ... BOUNDED ... CONVERGE ... INCREASING ... MUST ... SEQUENCE ...

AXIOM ____

g) The sequence $\{a_n\}$ is defined recursively as follows: $\begin{cases} a_1 = \frac{3}{2} \\ a_{n+1} = \sqrt{2a_n - \frac{1}{a_n}} \text{ for } n \geq 1 \end{cases}$.

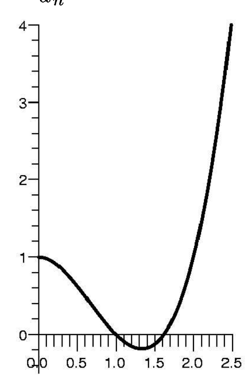
Explain why the sequence $\{a_n\}$ converges and find its limit <u>exactly</u>. You will need to use the previous parts of this problem.

Comment Here are the first 10 terms of the sequence, to 5 decimal place accuracy:

1.5, 1.52753, 1.54932, 1.56627, 1.57927.
1.58913, 1.59655, 1.60211, 1.60625, 1.60933

These numbers may help you emotionally but they should not be used in the proof.

To the right is the graph of $y = x^3 - 2x^2 + 1$.



(20) 10. a) Describe how to approximate $e^{\frac{1}{10}}$ with an error of size at most .0001 using a partial sum of a Taylor series. Give an explicit error estimate, and write an explicit partial sum which correctly approximates $e^{\frac{1}{10}}$. You may use the fact that $e < 3$. (Do the arithmetic needed for the error estimate; don't do the arithmetic involved in the partial sum!)
Comment A "best possible" partial sum is *not* requested, but the partial sum that's given should be supported by reasoning.
b) Find *some* interval of positive length with center at 0 so that $\cos x$ can be replaced by $1 - \frac{x^2}{2} + \frac{x^4}{24}$ with an error of size at most $\frac{1}{100}$ at any point of the interval. (Again, an error estimate is needed.)
Comment A "best possible" interval is *not* requested, but the interval that's given should be supported by reasoning.

(20) 11. One of the Bessel functions used to describe the vibration of a circular plate is defined by this infinite series: $J(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$.
a) Show that this series converges absolutely for all values of x.
b) Explain briefly why the result of a) implies that the series converges for all x.
c) Here are individual terms of the series for two values of x and for some values of n.

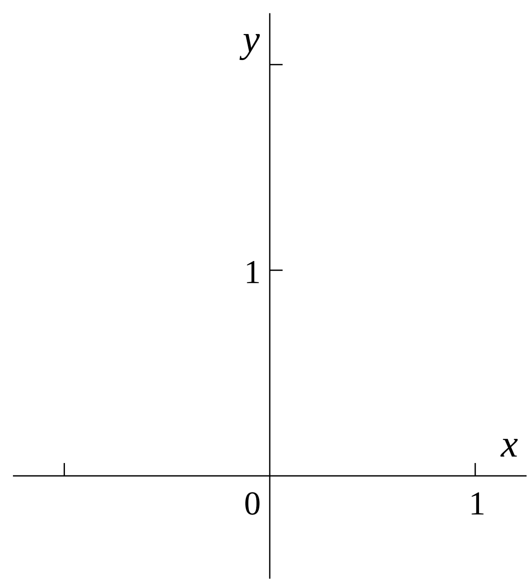
$\frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$x=1$	1	$-\frac{1}{4}$	$\frac{1}{64}$	$-\frac{1}{2{,}304}$	$\frac{1}{147{,}456}$	$-\frac{1}{14{,}745{,}600}$
$x=4$	1	-4	4	$-\frac{16}{9}$	$\frac{4}{9}$	$-\frac{16}{225}$

Use entries of this table and facts about the series to explain why $J(1)$ must be positive and $J(4)$ must be negative. You will need to discuss the "infinite tails" of each series separately and examine carefully how they affect the sums involved.

(16) 12. a) Sketch these polar curves on the axes given: the cardioid $r = 1 + \sin\theta$ and the circle $r = \sin\theta$.

b) Find the area which is inside the cardioid and outside the circle.

Comment Be *very* careful of the integration(s). Take advantage of symmetry, but the pictures, the areas, and the limits may not be related easily.



Final Exam for Math 192, section 1

December 21, 2005

NAME ____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes other than the distributed formula sheet may be used on this exam.
No calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	16	
3	16	
4	16	
5	16	
6	16	
7	16	
8	16	
9	20	
10	20	
11	20	
12	16	
Total Points Earned:		

Yes, the total number of points is 208. Please do the exam.