Counting the Fibonacci numbers

The Fibonacci numbers are the sequence $\{F_n\}$ defined recursively by the rule $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$ with the "initial conditions" $F_0 = 0$ and $F_1 = 1$. The sequence begins:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ 233,\ 377,\ 610,\ 987,\ 1597,\ 2584,\ 4181,\ 6765,\ \ldots$$

The F_n's occur in many aspects of computer science and other areas of applied mathematics. I'll describe a process which gets an explicit formula for these numbers and which also gives good asymptotic information about the size of the numbers. Both the process and the specific information obtained are useful in many applications. The process yields good information when used with other recursively defined sequences.

Stage 1: Constructing a power series

We build a power series out of the Fibonacci numbers:

$$0x^0 + 1x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 \ldots = \sum_{n=0}^{\infty} F_n x^n$$

which I'll call $S(x)$. The coefficient of x^n is the n^{th} Fibonacci number, F_n. Since $F_{n+2} = F_{n+1}+F_n$, I know that if I take $F_n x^n$ and multiply it by x^2 and add on $F_{n+1}x^{n+1}$ multiplied by x, I'll just get $F_{n+2}x^{n+2}$ when $\underline{n+2 \text{ is at least } 2}$. This means that if I take $S(x)$ and multiply it by x^2 and add on $S(x)$ multiplied by x, I'll get $S(x)$ again, except for the $n = 0$ and $n = 1$ terms. The $n = 0$ term is itself 0 and the $n = 1$ term is x, so what I get is:

$$xS(x) + x^2 S(x) = -x + S(x)\,.$$

I must subtract the x for the equation to be correct. We "solve for" $S(x)$:

$$S(x) = \frac{-x}{x^2 + x - 1}\,.$$

We can use this equation for $S(x)$ to get another description of the power series. Since $S(x)$ has exactly one power series centered at 0 (its Taylor series centered at 0), the alternative description is the formula I mentioned above.

Stage 2: The intervention of partial fractions

The roots of $x^2 + x - 1 = 0$ are $\frac{-1\pm\sqrt{5}}{2}$. I'll call r_+ the "plus root", $\frac{-1+\sqrt{5}}{2}$, and r_- the "minus root", $\frac{-1-\sqrt{5}}{2}$. Then $x^2 + x - 1 = (x - r_+)(x - r_-)$. We will now "rewrite" $\frac{-x}{x^2+x-1}$ using partial fractions:

$$S(x) = \frac{-x}{x^2 + x - 1} = \frac{-x}{(x - r_+)(x - r_-)} = \frac{A}{x - r_+} + \frac{B}{x - r_-}$$

A and B then must satisfy this equation: $-x = A(x - r_-) + B(x - r_+)$. If $x = r_+$, then $-r_+ = A(r_+ - r_-)$ but $r_+ - r_- = \sqrt{5}$, so $A = -\frac{r_+}{\sqrt{5}}$. If $x = r_-$, $-r_- = B(r_- - r_+)$ (notice $r_- - r_+$ here, not $r_+ - r_-$!) so $B = \frac{r_-}{\sqrt{5}}$. Therefore (factoring out the common $\frac{1}{\sqrt{5}}$):

$$S(x) = \frac{1}{\sqrt{5}}\left(-\frac{r_+}{x - r_+} + \frac{r_-}{x - r_-}\right)$$

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Stage 3: The sum of a geometric series

I'll change the "pieces" of $S(x)$ into something which looks like $\frac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$.

$$\frac{r_+}{x-r_+} = \frac{r_+ \cdot \frac{1}{r_+}}{(x-r_+)\cdot\frac{1}{r_+}} = \frac{1}{\frac{x}{r_+}-1} = \frac{-1}{1-\frac{x}{r_+}}$$

so that $a=-1$ and $r=\frac{x}{r_+}$. One more algebraic note:

$$\frac{1}{r_+} = \frac{1}{\frac{-1+\sqrt{5}}{2}} = \frac{2}{-1+\sqrt{5}} = \frac{2\cdot(-1-\sqrt{5})}{(-1+\sqrt{5})\cdot(-1-\sqrt{5})} = \frac{2(-1-\sqrt{5})}{1-5} = \frac{1+\sqrt{5}}{2} = -r_-$$

and therefore

$$\frac{r_+}{x-r_+} = \frac{-1}{1-\frac{x}{r_+}} = \frac{-1}{1-(-r_-x)} = -\sum_{n=0}^{\infty}(-r_-)^n x^n\,.$$

The other piece is similar, reversing the roles of r_- and r_+: $\frac{r_-}{x-r_-} = -\sum_{n=0}^{\infty}(-r_+)^n x^n\,.$

Now I put the sums back into $S(x)$. Care is needed since there are many $-$ signs around:

$$S(x) = \frac{1}{\sqrt{5}}\left(-\frac{r_+}{x-r_+}+\frac{r_-}{x-r_-}\right) = \frac{1}{\sqrt{5}}\left(- \; -\sum_{n=0}^{\infty}(-r_-)^n x^n + \; -\sum_{n=0}^{\infty}(-r_+)^n x^n\right)$$

Stage 4: The promised formula and asymptotics

The coefficient of x^n in the equation for $S(x)$ just above is: $\frac{1}{\sqrt{5}}\left((-r_-)^n-(-r_+)^n\right)$ so that this *must be* F_n. Here with values for r_- and r_+ is the anticipated

$$\textbf{EXPLICIT FORMULA}\text{: } F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

which is almost unbelievable because all the $\sqrt{5}$'s *must cancel* to produce an integer!

We can check the formula for $n=4$, which is already some work. I know

$$(a+b)^4 = a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

so $\left(\frac{1+\sqrt{5}}{2}\right)^4 = \frac{1^4+4\cdot1^3\cdot(\sqrt{5})^1+6\cdot1^2\cdot(\sqrt{5})^2+4\cdot1^1\cdot(\sqrt{5})^3+(\sqrt{5})^4}{2^4} = \frac{1+4\sqrt{5}+6\cdot5+4\cdot5\cdot\sqrt{5}+25}{16} = \frac{57+24\sqrt{5}}{16}$

and a similar computation shows that $\left(\frac{1-\sqrt{5}}{2}\right)^4 = \frac{57-24\sqrt{5}}{16}$ because the $\sqrt{5}$'s appear from the odd powers, so the minus signs give a minus sign exactly *there* in the result.

$$F_4 = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^4 - \left(\frac{1-\sqrt{5}}{2}\right)^4\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{57+24\sqrt{5}}{16}\right)-\left(\frac{57-24\sqrt{5}}{16}\right)\right) = \frac{1}{\sqrt{5}}\left(\frac{48\sqrt{5}}{16}\right) = 3\,(!).$$

The explicit formula above may be easier to understand if we use numerical approximations: with $\frac{1}{\sqrt{5}} \approx .447$ and $\frac{1+\sqrt{5}}{2} \approx 1.618$ and $\frac{1-\sqrt{5}}{2} \approx -.618$ we get

$$F_n \approx .447\left((1.618)^n - (-.618)^n\right)\,.$$

Notice $(-.618)^n \to 0$ rapidly as $n \to \infty$ (it is a power of a number whose absolute value is less than 1) so it is usually omitted. The result is a

$$\textbf{GOOD APPROXIMATION}\text{: } F_n \approx .447(1.618)^n\,.$$

The precise value of F_{20} is 6765 and this approximation gives 6758.9.

$S(x)$ is usually called the GENERATING FUNCTION of the sequence $\{F_n\}$.