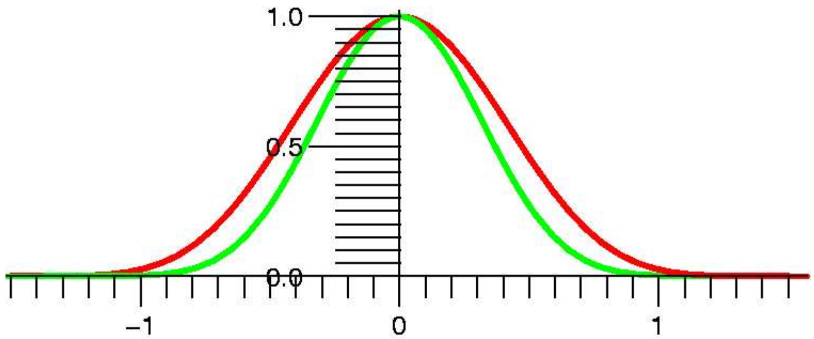
How to compute integrals of powers of cosine

Define I_n to be $\int_{-\pi/2}^{\pi/2}(\cos\theta)^n\,d\theta$.

What do we know? The whole sequence of numbers $\{I_n\}$ is a decreasing sequence. This is actually not too hard to see. Remember that I_n is the integral of $(\cos x)^n$ over the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. In this interval, the values of cosine are between 0 and 1. Therefore the integral will be less than the area of the box whose base is π long and whose height is 1. But, in fact, the higher the power, the smaller the value of the function except at those places where cosine is either 1 ($x=0$) or 0 ($x=\pm\frac{\pi}{2}$). Higher powers make smaller integrals. Here is a graph of two powers of cosine over the interval. The larger one is $(\cos x)^6$, with integral (approximately) .982, and the smaller one is $(\cos x)^{10}$ with integral (approximately) .773. The picture and the computations were both produced by `Maple`).



$(\cos x)^6$ and $(\cos x)^{10}$ on the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

We'll use integration by parts for the computation. The parts used aren't totally obvious, but they eventually yield a nice reduction formula.

$$\int_{-\pi/2}^{\pi/2}(\cos\theta)^n\,d\theta = (\cos\theta)^{n-1}\cdot\sin\theta\,\Big]_{-\pi/2}^{\pi/2} + (n-1)\int_{-\pi/2}^{\pi/2}(\cos\theta)^{n-2}(\sin\theta)^2\,d\theta$$

$$\int u\,dv \qquad = \qquad uv \qquad - \qquad \int v\,du$$

$$\left.\begin{array}{r} u=(\cos\theta)^{n-1} \\ dv=\cos\theta\,d\theta \end{array}\right\}\left\{\begin{array}{l} du=(n-1)(\cos\theta)^{n-2}(-\sin\theta)\,d\theta \\ \quad v=\sin\theta \end{array}\right.$$

Note first that we get $+$ in the actual application of integration by parts here because there are two minus signs, one from the integration by parts formula and one from the derivative of cosine. Also, the "penalty term" (what we pay for the privilege of exchanging the integrals – the stuff with the $]$) is actually 0 because $\sin\theta$ is 0 when θ is $\pm\pi/2$.

We can continue to massage the integral we've gotten by using $(\sin\theta)^2 = 1-(\cos\theta)^2$ and by dropping the term which equals 0. We then get:

$$\begin{aligned}\int_{-\pi/2}^{\pi/2}(\cos\theta)^n\,d\theta &= (n-1)\int_{-\pi/2}^{\pi/2}(\cos\theta)^{n-2}\left(1-(\cos\theta)^2\right)\,d\theta \\ &= (n-1)\int_{-\pi/2}^{\pi/2}(\cos\theta)^{n-2}\,d\theta - (n-1)\int_{-\pi/2}^{\pi/2}(\cos\theta)^n\,d\theta\end{aligned}$$

Now we can "solve" for c_n, the desired integral, by taking the second term on the right side of the equation to the left, adjusting the sign, and dividing by the resulting coefficient, n. We get:

$$\int_{-\pi/2}^{\pi/2} (\cos\theta)^n \, d\theta = \frac{n-1}{n} \int_{-\pi/2}^{\pi/2} (\cos\theta)^{n-2} \, d\theta$$

We can translate this:

$$I_n = \frac{n-1}{n} \, I_{n-2}$$

This reduction formula is valid if $n \geq 2$. We certainly know that $I_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^0 \, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx = \pi$ and $I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^1 \, dx = 2$. With these two values and the reduction formula we can exactly compute all of the I_n's. For example,

$$I_5 = \frac{4}{5} \cdot I_3 = \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 = \left(\frac{4 \cdot 2}{5 \cdot 3}\right) 2$$

and

$$I_6 = \frac{5}{6} \cdot I_4 = \frac{5}{6} \cdot \frac{3}{4} \cdot I_2 = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 = \left(\frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2}\right) \pi$$

These two computations provide examples of descriptions for I_n. The descriptions depend on the parity (even or odd) of n.

Computing π strangely: an unexpected reward

We know I_5 and I_6:

$$I_5 = \left(\frac{4 \cdot 2}{5 \cdot 3}\right) 2$$
$$I_6 = \left(\frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2}\right) \pi$$

And here are I_7 and I_8:

$$I_7 = \left(\frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3}\right) 2$$
$$I_8 = \left(\frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2}\right) \pi$$

We also know that the integrals of higher powers are smaller:

$$I_8 \leq I_7 \leq I_6$$

Let's look at the values of just these definite integrals:

$$\left(\frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2}\right) \pi \leq \left(\frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3}\right) 2 \leq \left(\frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2}\right) \pi$$

Now "simplify" the first two expressions, putting $\frac{\pi}{2}$ on the left-hand side:

UP $$\frac{\pi}{2} \leq \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}$$

Now "simplify" the last two expressions, putting $\frac{\pi}{2}$ on the right-hand side:

DOWN $$\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7} \leq \frac{\pi}{2}$$

Incidentally, if you don't believe this, `maple` reports that the overestimate of $\frac{\pi}{2}$ shown in **UP** is approximately 1.672, and the underestimate of $\frac{\pi}{2}$ shown in **DOWN** is approximately 1.463; note that 1.571 is a three-decimal place approximation of $\frac{\pi}{2}$.

The quotient of the **UP** overestimate divided by the **DOWN** underestimate is

$$\frac{\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7}}{\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7}} = \frac{8}{7}$$

We can extend this reasoning by looking at the formulas for I_{2n} and I_{2n+1} and I_{2n+2}. The discrepancy between the over– and underestimates will be $\frac{2n+2}{2n+1}$ which surely approaches 1 as n gets large. Therefore [!!!],

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

This "infinite product formula" is attributed to John Wallis (1616–1703). It converges very slowly to its limit. This isn't the formula used by the well-publicized computations which get billions of digits of π!

Some sample computations

I asked a silicon friend to compute some "partial products" (well, they're only *part* of the whole product). The command `mul` asks `Maple` to multiply out a formula over a given range. So, for example, `mul(j,j=1..5)` requests $1\cdot 2\cdot 3\cdot 4\cdot 5$, and returns 120, which is 5!, 5 factorial. Below I've organized the Wallis product in groups of two fractions at a time. So the first computation actually found the 400$^{\text{th}}$ partial product, and the third, which took a bit more than a tenth of a second, found the 20,000$^{\text{th}}$ partial product. Darn, the Wallis product converges very slowly. I need to put in the `evalf`'s because otherwise I'll just get a fraction of some really big integers.

```
> evalf(2*mul((2*j*2*j)/((2*j-1)*(2*j+1)),j=1..200));
                                   3.137677901
> evalf(2*mul((2*j*2*j)/((2*j-1)*(2*j+1)),j=1..1000));
                                   3.140807746
> evalf(2*mul((2*j*2*j)/((2*j-1)*(2*j+1)),j=1..10000));
                                   3.141514119
```

I honestly report that I needed four attempts to write the `Maple` instructions correctly!