October 3, 1993

Answers to Exam 1 of Math 192

Here are answers (together with the questions!) to the exam. I have written *my* answers. There are frequently other ways to do these problems. In smaller type I have included a number of remarks not strictly necessary for the solution of the problems but perhaps valuable for strategic considerations.

(12) 1. Suppose $V(x) = 3\arcsin(2x)$

a) What is the domain of V? (Your answer should be an interval.)

You should know arcsin well; exploit this knowledge.

The function arcsin has domain $[-1,1]$. The effect of the 2 is to double *inputs* to arcsin. So legal inputs to V will have half the magnitude of legal inputs to arcsin. So the domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

b) What is the range of V? (Your answer should be an interval.)

The range, the collection of *outputs*, is triple that of arcsin. Thus the range is $\left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$.

c) Give an explicit formula for the function inverse to V.

If $y = V(x)$ then $y = 3\arcsin(2x)$, so $\frac{y}{3} = \arcsin(2x)$ and $\sin\left(\frac{y}{3}\right) = 2x$ and so (finally) $\frac{1}{2}\sin\left(\frac{y}{3}\right) = x$. The function $W(m) = \frac{1}{2}\sin\left(\frac{m}{3}\right)$ is the function inverse to V. Here the specific variable in the function does not matter: it is a **dummy** variable and describes what the function does to numbers.

d) Compute $V\left(\frac{1}{2}\sin\left(\frac{7\pi}{4}\right)\right)$.

Tricky. $V\left(\frac{1}{2}\sin\left(\frac{7\pi}{4}\right)\right) = 3\arcsin\left(2\cdot\frac{1}{2}\sin\left(\frac{7\pi}{4}\right)\right) = 3\arcsin\left(\sin\left(\frac{7\pi}{4}\right)\right) =$ (since sin is periodic with period 2π) $3\arcsin\left(\sin\left(\frac{7\pi}{4} - 2\pi\right)\right) = 3\arcsin\left(\sin\left(-\frac{\pi}{4}\right)\right) =$ (since we're now in the domain where arcsin <u>is</u> the inverse to sin!) $3\cdot\left(-\frac{\pi}{4}\right) = -\frac{3\pi}{4}$.

(16) 2. Compute $\dfrac{dy}{dx}$: (Do <u>not</u> simplify your answers!)

a) & b) are routine. c) was contributed by the Marquis du Sade.

a) $y = \arctan\left(x^3+7\right)$

$$y' = \frac{1}{1+(x^3+7)^2}\cdot\left(3x^2\right).$$

b) $y = \ln\left(\dfrac{x^2+1}{x+2}\right)$

$$y' = \frac{1}{\left(\frac{x^2+1}{x+2}\right)}\cdot\left(\frac{(2x)(x+2)-(1)(x^2+1)}{(x+2)^2}\right)$$

c) $y = 7x^{\left(x^x\right)} + 5\left(x^x\right)^x$

Whew! Let us try to think clearly, always animated by the wonderful formula (the definition!): $A^B = e^{B\ln A}$. Then we can take apart the pieces of y:

$x^{\left(x^x\right)} = e^{\left(x^x\right)\ln x} = e^{e^{x\ln x}\ln x}$

and

$\left(x^x\right)^x = \left(e^{x\ln x}\right)^x = e^{x(x\ln x)} = e^{x^2\ln x}$.

Now it's "easy". Just write: $y = 7x^{\left(x^x\right)} + 5\left(x^x\right)^x = 7e^{e^{x\ln x}\ln x} + 5e^{x^2\ln x}$ and turn the chain rule loose on it. We'll get:

$$y' = 7e^{e^{x\ln x}\ln x}\cdot\left(\left(e^{x\ln x}\cdot\left(1\cdot\ln x + x\cdot\tfrac{1}{x}\right)\right)\ln x + \left(e^{x\ln x}\cdot\tfrac{1}{x}\right)\right) + 5e^{x^2\ln x}\cdot\left(2x\cdot\ln x + x^2\cdot\tfrac{1}{x}\right)$$

(18) 3. Perform the indicated integrations. Give explicit numerical answers but leave such constants as $\sqrt{7}$ and π without further approximation.

Again, this is mostly routine.

a) $\displaystyle\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx$

Substitute $x^2 = \xi$ (choice of letters either to increase awareness or simply to annoy!). Then since $2x\,dx = d\xi$ so $x\,dx = \frac{1}{2}\,d\xi$, we get:

$$\int \frac{x}{\sqrt{1-x^4}}\,dx = \int \frac{1}{2}\frac{1}{\sqrt{1-\xi^2}}\,d\xi = \frac{1}{2}\arcsin(\xi) + C = \frac{1}{2}\arcsin\left(x^2\right) + C.$$

The definite integral becomes:

$$\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\arcsin\left(x^2\right)\Big]_1^{\sqrt{2}} = \frac{1}{2}\left(\arcsin\left(\frac{1}{2}\right) - \arcsin(0)\right) = \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{12}.$$

b) $\displaystyle\int_1^2 \frac{dx}{\sqrt{x}+x}$

If $\eta = \sqrt{x}$, problems will just disappear! Indeed, it should look familiar because it was a homework problem. It is a bit easier to compute if we rewrite the relationship as $\eta^2 = x$, for then $2\eta\,d\eta = dx$ and the integral changes in the following way:

$$\int \frac{dx}{\sqrt{x}+x} = \int \frac{2\eta\,d\eta}{\eta + (\eta)^2} = 2\int \frac{d\eta}{1+\eta} = 2\ln(1+\eta) + C = 2\ln(1+\sqrt{x}) + C.$$

The definite integral becomes: $\displaystyle\int_1^2 \frac{dx}{\sqrt{x}+x} = 2\ln(1+\sqrt{x})\Big]_1^2 = 2\ln(1+\sqrt{2}) - 2\ln(1+\sqrt{1}) = \ln\left(\frac{3+2\sqrt{2}}{4}\right).$

The last entry is necessary only if you are a member of the International Association of Quasi-simplification (IAQ).

c) $\displaystyle\int_0^{\pi/3} \tan x\,(\sec x)^2\,dx$

Well, the derivative of $\tan x$ is $(\sec x)^2$ so this integral is just $\displaystyle\frac{1}{2}(\tan x)^2\Big]_0^{\frac{\pi}{3}} = \frac{1}{2}\left(\tan\left(\frac{\pi}{3}\right)\right)^2 - \frac{1}{2}0^2 = \frac{1}{2}\left(\sqrt{3}\right)^2 = \frac{3}{2}.$

(18) 4. Consider the function $G(x) = \ln(1+x^2)$.

Before we begin the computations, note that if x is any real number, $1 + x^2 \geq 1$, so the **domain** of G is all real numbers.

a) What is $\lim_{x\to\infty} G(x)$?

When w is a large positive number, $\ln w$ is a large positive number. Therefore $\lim_{x\to\infty} G(x) = +\infty$.

b) What is $\lim_{x\to-\infty} G(x)$?

Note that $G(-x) = G(x)$ (this also has consequences for the graph of G, which will therefore be symmetric with respect to the y–axis). But then $\lim_{x\to-\infty} G(x) = +\infty$ by using the result of the previous part of the problem.

c) Compute $G'(x)$ carefully. Where is $G'(x) = 0$?

$G'(x) = \dfrac{2x}{1+x^2}$. Thus the only critical points of G are where $2x = 0$, that is, where $x = 0$. But $G(0) = \ln(1+0^2) = \ln(1) = 0$. So $(0,0)$ is the only critical point of G. The *sign* of G is the same as the sign of $2x$, since the denominator, $1+x^2$, is always positive. Thus G is increasing in $[0,+\infty)$ and decreasing in $(-\infty,0]$.

d) Compute $G''(x)$ carefully. Where is $G''(x) = 0$?

$G''(x) = \dfrac{(2)\cdot(1+x^2) - (2x)\cdot(2x)}{(1+x^2)^2} = \dfrac{2+2x^2-4x^2}{(1+x^2)^2} = \dfrac{2-2x^2}{(1+x^2)^2}$, which means the only possible inflection points are where $2 - 2x^2 = 0$. That's when $x = \pm 1$. Of course $G(\pm 1) = \ln(1+(\pm 1)^2) = \ln(2)$. In fact, G'' *does* change sign at $x = \pm 1$, so the two points $(\pm 1, \ln 2)$* are points of inflection (P.O.I.'s)**. The concavity is *up* for $-1 < x < 1$ (check the sign of G''!) and *down* for $-\infty < x < -1$ and $1 < x < \infty$.

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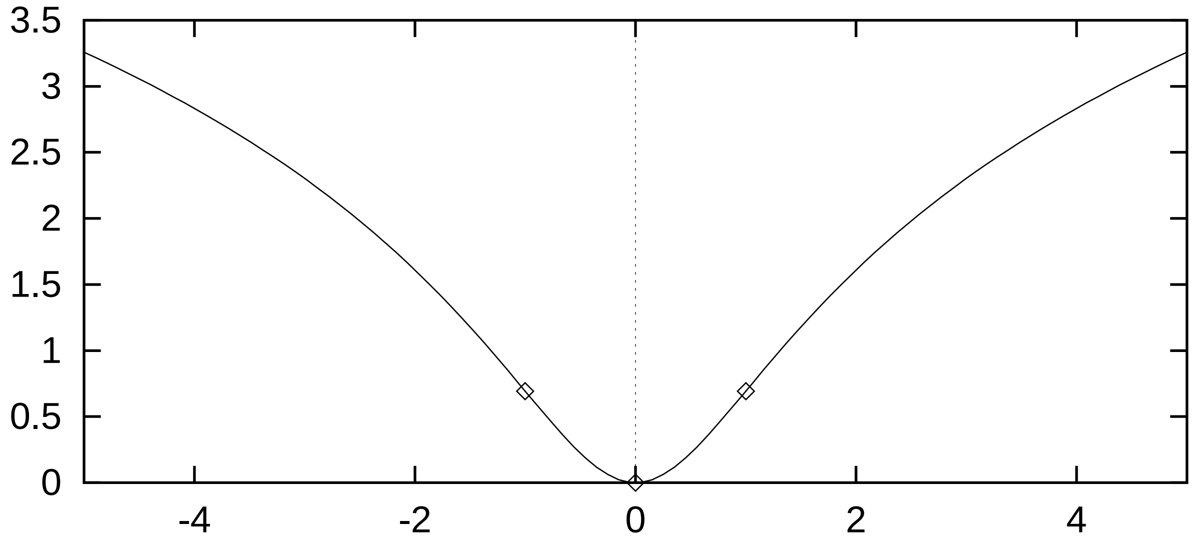
* Yes, yes. I know that ln 2 is (approximately) 0.69314 71805 59945 30941 72321 but I don't need this to complete the problem.

** Poi: "A native Hawaiian food prepared from the taro root pounded to a paste and allowed to ferment."

e) Use all of the information above to sketch the graph of $y = G(x)$. **Be sure to indicate explicitly** any local extrema or inflection points, all regions of increase and decrease, and all regions of concavity.

$(0, 0)$ is a local (and, in fact, global) minimum. The other information asked for has already been given. Only the graph remains, and here it is:



Of course when $|x|$ is very large, we really "know" the shape of this graph, since then $G(x) = \ln(1 + x^2) \approx \ln(x^2) = 2\ln|x|$, and the graph just looks like $y = 2\ln|x|$.

(12) 5. A small meteorite falls into the rapidly flowing Raritan River. One hour later, the meteorite is 300°C hotter than the river. One hour after that measurement, the meteorite is 100°C hotter than the river. How much hotter than the river was the meteorite when it first hit? (Assume that the rate of temperature decrease is directly proportional to the difference in the temperatures of the river and the meteorite.)

Aliens come out of the meteorite and take over the buses shuttling around the Rutgers–NB campuses. No one notices.

Suppose $T(t)$ is the **difference** in temperature between the meteorite and the river at time t. Here T is measured in degrees C and t is measured in hours from the meteorite's impact. Then we know: $T(1) = 300$ and $T(2) = 100$. Of course, the sentence beginning with the word "assume" is Newton's law of cooling, and in this case it means $T' = kT$ which in turn implies that $T(t) = Ce^{kt}$. We want to find C, which is $T(0)$. Now we know $Ce^k = 300$ and $Ce^{2k} = 100$. There are various ways to proceed here, but what if we just divide the second equation by the first? Then we get $e^k = \frac{1}{3}$, so $k = -\ln 3$. Substituting this into the first equation yields: $C\exp(-\ln 3) = 300$, which, after juggling a tiny bit, gives $C = 900$. This prompts the answer: the meteorite was 900°C hotter than the river when it hit.

(24) 6. Integrate:

I hope nothing here is really unexpected.

a) $\int x^2 \cos(ax)\,dx$ ("a" is a constant here.) Integrate by parts twice, reducing the degree each time.

$$\int x^2\cos(ax)\,dx = x^2\cdot\frac{1}{a}\sin(ax) - \int \frac{1}{a}\sin(ax)2x\,dx$$

$$\int u\,dv \qquad = \qquad uv \quad - \quad \int v\,du$$

$$\left.\begin{array}{r} u = x^2 \\ dv = \cos(ax)\,dx \end{array}\right\}\left\{\begin{array}{l} du = 2x\,dx \\ v = \frac{1}{a}\sin(ax) \end{array}\right.$$

But $-\int \frac{1}{a}\sin(ax)2x\,dx = -\frac{2}{a}\int x\sin(ax)\,dx$. and we can integrate *that* by parts:

$$\int x\sin(ax)\,dx = x\cdot -\frac{1}{a}\cos(ax) - \int -\frac{1}{a}\cos(ax)\,dx$$

$$\int u\,dv \qquad = \qquad uv \quad - \quad \int v\,du$$

$$\left.\begin{array}{r} u = x \\ dv = \sin(ax)\,dx \end{array}\right\}\left\{\begin{array}{l} du = dx \\ v = -\frac{1}{a}\cos(ax) \end{array}\right.$$

The final integral is easy: $\int -\frac{1}{a}\cos(ax)\,dx = -\frac{1}{a}\int \cos(ax)\,dx = -\frac{1}{a^2}\sin(ax) + C$.

Now I will try to pack them all together correctly:

$$\int x^2\cos(ax)\,dx = x^2\cdot\left(\frac{1}{a}\right)\sin(ax) - \left(\frac{2}{a}\right)\cdot\left(x\cdot -\frac{1}{a}\cos(ax) - \left(-\frac{1}{a^2}\sin(ax)\right)\right) + C$$

b) $\int \frac{dx}{4+9x^2}$

Try the random? substitution $3x = 2\tan\beta$. Then $4+9x^2 = 4+(3x)^2 = 4+4(\tan\beta)^2 = 4(1+(\tan\beta)^2) = 4(\sec\beta)^2$. Also, $3\,dx = 2(\sec\beta)^2\,d\beta$, so the integral changes what coincidences! in the following fashion:

$\int \frac{dx}{4+9x^2} = \int \frac{2}{3}\cdot\frac{(\sec\beta)^2}{4(\sec\beta)^2}\,d\beta = \frac{1}{6}\beta + C$. Since $\beta = \arctan\left(\frac{3x}{2}\right)$ our answer is $\frac{1}{6}\arctan\left(\frac{3x}{2}\right) + C$.

c) $\int \frac{dx}{\sqrt{x^2-2x-3}}$ This is just completing the □, of course, followed by an appropriate trig substitution.

$x^2-2x-3 = x^2-2x+1-1-3 = (x-1)^2-4$. Clearly Clearly? Yeah, sure. the substitution $x-1 = 2\sec\gamma$ will be efficacious. Well, the "2" is motivated by the "4" and the use of sec is desired because we have the square root of {something squared minus a positive constant}. Then $(x-1)^2-4 = 4(\sec\gamma)^2-4 = 4\left((\sec\gamma)^2-1\right) = 4(\tan\gamma)^2$ and $dx = 2\sec\gamma\tan\gamma\,d\gamma$, so the integral becomes:

$$\int \frac{dx}{\sqrt{x^2-2x-3}} = \int \frac{2\sec\gamma\tan\gamma}{2\tan\gamma}\,d\gamma = \int \sec\gamma\,d\gamma = \ln|\sec\gamma+\tan\gamma| + C.$$

But $\frac{x-1}{2} = \sec\gamma$ and $\tan\gamma = \sqrt{(\sec\gamma)^2-1} = \sqrt{\left(\frac{x-1}{2}\right)^2 - 1}$ Yes, there are better algebraic ways of writing it, but I just want to get it done! and thus the final answer is: $\ln|\sec\gamma+\tan\gamma| + C = \ln\left|\frac{x-1}{2} + \sqrt{\left(\frac{x-1}{2}\right)^2-1}\right| + C$.

d) $\int e^{3x}\sin 5x\,dx$ Integrate by parts twice, & solve for the "unknown" integral.

Suppose $I = \int e^{3x}\sin 5x\,dx$. Then use $\left.\begin{array}{r} u = e^{3x} \\ dv = \sin 5x\,dx \end{array}\right\}\left\{\begin{array}{l} du = 3e^{3x}\,dx \\ v = -\frac{1}{5}\cos 5x \end{array}\right.$ in $\int u\,dv = uv - \int v\,du$ to obtain:

$I = -\frac{1}{5}e^{3x}\cos 5x + \frac{3}{5}\int e^{3x}\cos 5x\,dx$. Yes, I combined two minuses there. I'm tired of typing – this is the last problem, after all. Now integrate by parts again, applying the method to the integral $J = \int e^{3x}\cos 5x\,dx$. Here the choice of parts *is* quite important, since the "wrong" choice can lead to the rather frustrating equation $0 = 0$. But if you look ahead a bit you can see what choice to make – you want another $\frac{1}{5}$ coming out. So try:

$\left.\begin{array}{r} u = e^{3x} \\ dv = \cos 5x\,dx \end{array}\right\}\left\{\begin{array}{l} du = 3e^{3x}\,dx \\ v = \frac{1}{5}\sin 5x \end{array}\right.$ and get $J = \frac{1}{5}e^{3x}\sin 5x - \frac{3}{5}\int e^{3x}\sin 5x\,dx$. Stuffing it all back together we get the super equation: $I = -\frac{1}{5}e^{3x}\cos 5x + \frac{3}{5}\left(\frac{1}{5}e^{3x}\sin 5x - \frac{3}{5}I\right) = -\frac{1}{5}e^{3x}\cos 5x + \frac{3}{25}e^{3x}\sin 5x - \frac{9}{25}I$ or $\frac{34}{25}I = -\frac{1}{5}e^{3x}\cos 5x + \frac{3}{25}e^{3x}\sin 5x$ so $I = \frac{25}{34}\left(-\frac{1}{5}e^{3x}\cos 5x + \frac{3}{25}e^{3x}\sin 5x\right)$ oh yeah $+C$.