Review problems for 192:01

Here are some review problems for Math 192, mostly written by a colleague for a previous instantiation of the course.

1. Find the interval of convergence for each of the following power series (that is, for each power series, find all x's where the series converges). Justify your answers.

$$\text{A. } \sum_{n=0}^{\infty} \frac{nx^n}{3^n} \qquad \text{B. } \sum_{n=1}^{\infty} \frac{x^n}{5n^{2/3}}$$

2. For each of the following, find $\lim_{n\to\infty} a_n$ if the **sequence** $\{a_n\}$ converges, or explain why it does not converge. A. $a_n = \dfrac{\ln n}{n^3}$ B. $a_n = (-1)^n\left(1 - \dfrac{1}{n}\right)$

3. Determine whether each of the following series converges absolutely, converges conditionally, or diverges. Justify your answer by applying an appropriate test or theorem.

$$\text{A. } \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^{1/2}} \quad \text{B. } \sum_{n=1}^{\infty} \frac{n}{n^3+1} \quad \text{C. } \sum_{n=1}^{\infty} \frac{\ln n}{n^3} \quad \text{D. } \sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right) \quad \text{E. } \sum_{n=0}^{\infty} \frac{10^n}{n!}$$

4. A. Sketch the graph of $y = \ln x$ and fill in the blanks: The domain of this function is ____________. The range of this function is ____________.

B. Sketch the graph of $y = e^{2x}\sinh x$ and fill in the blanks. The domain of this function is ____________. The range of this function is ____________.

5. Compute the following integrals.

$$\text{A. } \int \frac{3x^2}{\sqrt{x^3+1}}\,dx \qquad \text{B. } \int \sin^3(2x)\,dx \qquad \text{C. } \int \sec^4 x\,dx$$

$$\text{D. } \int \frac{dx}{x(x^2+1)} \qquad \text{E. } \int \frac{dx}{\sqrt{3+2x-x^2}} \qquad \text{F. } \int x\ln x\,dx$$

6. Determine whether each of the following converges or not. If the integral converges, compute it if possible, and otherwise give good upper and lower bounds for its value (that is, find a and b with $a >$ value of integral $> b$).

$$\text{A. } \int_0^{1/4} \frac{dx}{x\ln x} \qquad \text{B. } \int_0^{\infty} \frac{dx}{1+x^2} \qquad \text{C. } \int_0^1 \frac{1+2^x}{\sqrt{x}}\,dx$$

7. Find $\dfrac{dy}{dx}$ if A. $y = \sinh x + 2^x$ B. $y = \ln(\arctan x)$ C. $y = \sec(e^x)$

8. Sketch the graphs of the following equations. Here r and θ are in **polar coordinates**.

$$\text{A. } r = 2(1+\sin\theta) \qquad \text{B. } r^2 = \cos(2\theta)$$

9. Write an integral which gives the area inside the curve $r = 2\,(1+\sin\theta)$ and then **compute** this integral. (Note that this is one of the functions you graphed in the preceding problem.)

10. Throughout this problem $f(x)$ will denote the function $f(x) = \ln(1 + x)$.

A. Find the Taylor polynomial $p_3(x)$ of order three for $f(x)$ with center at $a = 0$.

B. Write the formula for the remainder term $R_3(x)$.

C. Assuming that $0 < x < 1/2$, estimate $R_3(x)$.

11. A. Write the Taylor series with center at $a = 0$ for e^t.

B. Using the series in A, write the Taylor series with center at $a = 0$ for $e^{(x^2)}$.

C. By integrating the series in B, express $\displaystyle\int_0^1 e^{(x^2)}\,dx$ as an infinite series.

12. A. Write the Taylor series with center at $a = 0$ for $\sin x$ and $\cos x$.

B. Give the first three non-zero terms in the Taylor series with center at $a = 0$ for $f(x) = (x\sin x) - 2(1 - \cos x)$.

C. Find $\displaystyle\lim_{x\to 0} \frac{x\sin x - x^2}{4x^4}$.

13. A parametric curve is defined by the equations $\begin{cases} x(t) = t\ln t \\ y(t) = 4(t + \cos t) \end{cases}$ for $0 < t < \infty$.

A. Find $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

B. Find the slope of this curve in the (x, y) plane at the point on the curve corresponding to $t = 1$. What is the equation of the line tangent to the curve at this point?

C. Find the points in the (x, y) plane at which this curve has a horizontal tangent line.

D. Find the points in the (x, y) plane at which this curve has a vertical tangent line.

14. The population of a certain species changes at a rate proportional to the **square** of the current population. An initial population of 1 is observed two years later to have doubled. What is the population one year after that observation? After an additional year? (Note: populations are measured in units of 10^6 individuals, but that is not relevant to the problem.)

15. Consider the power series $\displaystyle\sum_{n=1}^{\infty}(n^2 + 3n + 7)x^n$.

a) What is the radius of convergence of this series?

b) Inside its radius of convergence, the sum of this series is a rational function (a quotient of polynomials). What is this function?

Comment You may want to break up the series into several pieces and handle each piece separately. The answer is not necessarily "pretty".

16. What is $\displaystyle\lim_{x\to 0} \frac{\arctan(x^2) - (\arctan x)^2}{x^4}$? Don't use l'Hôpital's rule four times! Please don't.