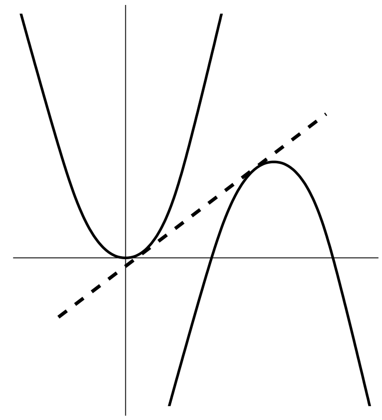
#1 **Problems for 192:01** 8/30/2005

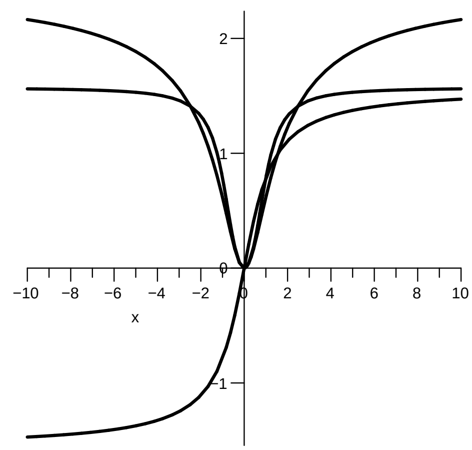
Please write solutions to two of these problems. Solutions are due on Wednesday, September 6. Some information on what's expected is on the reverse of this page.

1. Consider the parabola $y = x^2$. Flip it and move it right, to create a parabola opening "down" and intersecting the x-axis at $x = 1$ and $x = 2$. An equation for such a parabola is $y = -(x-1)(x-2)$. To the right is a rough sketch of the two parabolas and a straight line tangent to both of the parabolas. Find an equation for that line. Include *any* other information (labeled pictures) that you think is useful.



Comment Sometimes my problems, perhaps imitating "real life", may be ... deceptively stated. Or even confusing. So part of the problem is making sense out of the problem. Also this becomes a good excuse for sloppiness.

2. a) The picture here was drawn by `maple`. The style shown is *unconstrained*: "Unconstrained scaling adjusts the axes' scale to fit the plot window." (The graph is stretched to fill a square.) The functions with displayed graphs are $\arctan x$, $\arctan(x^2)$, and $(\arctan x)^2$. Match the curves with the functions. Find the coordinates of all intersections of the curves, exactly when possible, numerically otherwise. Explain your conclusions.



b) Suppose two functions are continuous and strictly increasing on an interval. Suppose they intersect a finite number of times. How many times can they intersect? Support your answer with an explanation or appropriate examples.

3. Suppose the following is known about the function F, whose domain is all of $\mathbb{R}$:

i) $F(0) = 2$; ii) $F'(x) = x\sqrt{x^6+1}$.

A consequence of the Mean Value Theorem is that the values of F are uniquely specified by these conditions. (You need not prove this!)

a) Graph $y = F(x)$ as well as you can on the interval $[-1, 2]$.

Where in this interval is the function increasing or decreasing? What critical points are there? How big (small?) can the function be at -1 and 2?

b) Find a Riemann sum for $\int_{-1}^{2} F(x)\,dx$ which is within 10^{-50} of the correct value.

Note You are not asked to compute this Riemann sum. You should write the sum and justify your assertion.

4. Suppose f is a function defined by $f(x) = \int_0^x \frac{1}{\sqrt{1+t^3}}\,dt$ for $x > -1$ and g is the function inverse to f. Then g satisfies a differential equation of the following form: $g''(x) = kg(x)^2$ where k is a constant. What is k?

OVER

This is the initial sheet of *workshop problems* for Math 192. These problems may be more challenging and interesting than most textbook problems.

Include in your solutions any information, including pictures (properly labeled!) and computations, that you think is useful. Your written solutions should be of high-quality, with the explanation of your solutions to problems given in complete English sentences. You will be graded on presentation as well as on mathematical content. Neatness counts (use staples or paperclips!). While I encourage you to discuss the problem with other students and with me (either via e-mail or in person), the written work you hand in must be your own. What follows is a sample problem and its solution, with a few comments.

An example of a workshop problem solution

Here is a problem that was a candidate for your first assignment, mostly because it used the Mean Value Theorem. The small print afterwards was part of the problem statement.

> **Problem Statement** Suppose that $f'(x) = \frac{2}{1+x^4} - \frac{3}{4+x^4}$. Is $f(0) < f(1)$?
> Please note: It is not likely at this time that you can write a formula for a function with this derivative (and, by the way, such a formula wouldn't really help very much!). So you will have to make some indirect argument, just using the information you have about the derivative.

Comments The solution should be written in COMPLETE ENGLISH SENTENCES. Details of routine computations generally don't need to be shown, unless (as here!) the logic involved is important. Solutions don't need to be long (think of them as brief technical reports) but they must be understandable and correct.

> **Solution** The answer is "Yes, $f(0) < f(1)$." Here is an explanation. One consequence of the Mean Value Theorem is the following result:
>
> > Suppose a differentiable function has a positive derivative on an interval. Then the function is strictly increasing on that interval.
>
> We therefore will know that $f(0) < f(1)$ if we can verify that $\frac{2}{1+x^4} - \frac{3}{4+x^4} > 0$ for x in $[0, 1]$. This inequality is equivalent to $\frac{2}{1+x^4} > \frac{3}{4+x^4}$. Since for x in $[0, 1]$, both $1 + x^4$ and $4 + x^4$ are positive, this inequality is equivalent to $2(4 + x^4) > 3(1 + x^4)$ which simplifies to $5 > x^4$. This last inequality is certainly correct if x is in $[0, 1]$.

More comments Solutions need *not* be typed, but multiple pages should be numbered. If you hand in more than one problem, please begin each problem on a separate page. Multiple pages should be attached to each other with a paperclip or a staple.

$\longrightarrow$ Neatness, correctness, and ease of reading are important. $\longleftarrow$

By the way, a silicon friend found this function whose derivative is given by the formula in the problem statement:

$$\frac{\sqrt{2}}{4}\ln\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}x-1)$$
$$+\frac{3}{16}\ln(x^2-2x+2) - \frac{3}{8}\arctan(x-1) - \frac{3}{16}\ln(x^2+2x+2) - \frac{3}{8}\arctan(x+1)$$

Does knowing this formula help or is studying the derivative easier?