Students may work individually or in groups of sizes two or three. Individuals and pairs of students should write solutions to two of these problems, while "triples" should hand in solutions to three of them. Solutions are due on Monday, October 3.

Solutions will be graded both for mathematical content and for exposition, which should be in **complete English sentences**. You don't need to show every computational detail, but please give enough information to allow readers to follow your reasoning. You may include appropriate diagrams, which should be carefully drawn and labeled.

Students who work in teams should be aware that each of them is responsible for *all work* the team hands in. I therefore suggest, at least, that all members of a team should read everything handed in by their team.

1. a) For x near 0, $\sin x$ is well-approximated by its tangent line at $x = 0$. What is this tangent line?

b) For many purposes, approximation over an interval is preferred over approximation near a point. One criterion for assessing the accuracy of such an approximation is *mean-square error*. The mean-square error between a straight line $y = Ax$ going through the origin and the function $\sin x$ over the interval $[0, 1]$ is given by the definite integral $\int_0^1 (\sin x - Ax)^2\, dx$. Find the A which minimizes this integral.
Hint Expand the integrand, compute the integral, and then minimize the function of A.

c) Sketch $\sin x$ and the straight lines found in a) and b) on the unit interval $[0, 1]$.

2. Consider the function $F(x) = \frac{e^{Ax}}{1+e^{Ax}}$ where A is a constant.

a) Sketch $y = F(x)$ for x in $[-1, 1]$ if $A = 10^{10}$. Explain why your drawing is correct.

b) Sketch $y = F(x)$ for x in $[-1, 1]$ if $A = 10^{-10}$. Explain why your drawing is correct.

c) Sketch $y = F(x)$ for x in $[-1, 1]$ if $A = -10^{10}$. Explain why your drawing is correct.

d) Sketch $y = F(x)$ for x in $[-1, 1]$ if $A = -10^{-10}$. Explain why your drawing is correct.

3. Very recently, Andrew Wiles *proved* that there were no solutions to the Fermat equation $a^n + b^n = c^n$ if a, b, c, and n are positive integers, with $n > 2$. (There are, of course solutions when $n = 2$: for example, $3^2 + 4^2 = 5^2$.)

a) Does the equation $4^x + 5^x = 6^x$ have a solution?* If it does, explain why it has a solution and find an approximate solution with accuracy $\pm .001$. If it does not, explain why.

b) Suppose a, b, and c are positive real numbers. Explore whether the equation $a^x + b^x = c^x$ must have a solution.
Comment This is a free form question: try to answer it as well as you can. You are not asked to provide a formula for x, you are merely asked to find conditions which will guarantee that such an x either does or does not exist.

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* There is no mention of the word "integer" in this question!

4. a) Compute $\int_1^2 \frac{1}{x^2}\,dx$.

b) Suppose α is a small positive number. Compute $I_\alpha = \int_1^2 \frac{1}{x^2+\alpha}\,dx$. What is $\lim_{\alpha\to 0^+} I_\alpha$? (You *must* compute this from your final formula for I_α, not just plug $\alpha = 0$ into the integral. To see why, do problem 5.)

c) Suppose β is a small positive number. Compute $J_\beta = \int_1^2 \frac{1}{x^2-\beta}\,dx$. What is $\lim_{\beta\to 0^+} J_\beta$? (You *must* compute this from your final formula for I_β, not just plug $\beta = 0$ into the integral. To see why, do problem 5.)

d) Draw some pictures of the areas computed by I_α and J_β (for small α and β) and the area computed in a). Do these help to explain any coincidences?

e) Is the function $F(t) = \int_1^2 \frac{1}{x^2+t}\,dx$ (defined for $t > -1$) continuous at $t = 0$? Use the definition of continuity (on page 124 of the textbook) and your answers for a), b), and c).

5. Suppose $F_A(x) = A^2(1-x)x^A$. Here A is a positive real number .

a) Sketch $F_1(x)$, $F_{10}(x)$, and $F_{10^{10}}(x)$ for x in $[0,1]$.

b) Suppose $g(A) = \int_0^1 F_A(x)\,dx$. Compute $g(A)$. What is $\lim_{A\to\infty} g(A)$?

c) Consider x in $[0,1]$ only and let $H(x) = \lim_{A\to\infty} F_A(x)$. Compute $H(x)$ and $\int_0^1 H(x)\,dx$.

d) Parts b) and c) compare $\lim_{A\to\infty} \int_0^1 F_A(x)\,dx$ and $\int_0^1 \lim_{A\to\infty} F_A(x)\,dx$. Do the pictures in part a) help you understand the results?