Students may work individually or in groups of sizes two or three. Individuals and pairs of students should write solutions to two of these problems, while "triples" should hand in solutions to three of them. Solutions are due on Monday, October 31.

These problems are intended to be challenging, and therefore students should feel comfortable consulting the instructor about them.

Solutions will be graded both for mathematical content and for exposition, which should be in **complete English sentences**. You don't need to show every computational detail, but please give enough information to allow readers to follow your reasoning. You may include appropriate diagrams, which should be carefully drawn and labeled.

Students who work in teams should be aware that each of them is responsible for *all work* the team hands in. I therefore suggest, at least, that all members of a team should read everything handed in by their team.

1. a) Graph $\ln x$ and x on the interval $[2, 10]$.

b) Prove with calculus that $x > \ln x$ on the interval $[2, \infty)$.

c) Show that $\int_2^\infty \frac{1}{\ln x}\,dx$ diverges.

d) Does $\int_2^\infty \frac{1}{(\ln x)^2}\,dx$ converge? Why?

Comment Analysis of d) does *not* follow from a), b) and c). There's a similar approach which you can discover. Hints are available.

2. Test each of the following integrals for convergence or divergence, showing the method used. Note: there is more than one thing to worry about in each of these integrals, and none of them can be evaluated in any simple form.

a) $\int_0^2 \frac{e^{\cos x}}{\sqrt{x^3(2-x)}}\,dx$

b) $\int_0^2 \frac{e^{\cos x}}{\sqrt{x(2-x)}}\,dx$

c) $\int_2^\infty \frac{e^{\cos x}}{\sqrt{x(x-2)}}\,dx$

d) $\int_2^\infty \frac{e^{\cos x}}{\sqrt{x^3(x-2)}}\,dx$

3. The integral $\int_1^\infty \frac{e^{-x^3}}{1+x^2}\,dx$ certainly converges. One simple way to try to approximate its value is to write it as the sum of two integrals:

$$\int_1^\infty \frac{e^{-x^3}}{1+x^2}\,dx = \int_1^T \frac{e^{-x^3}}{1+x^2}\,dx + \int_T^\infty \frac{e^{-x^3}}{1+x^2}\,dx$$

where T is some large number. Then any of our numerical techniques could be used to approximate the first (proper!) definite integral on the right-hand side of the equation. We could just *neglect* the second integral if we knew it was small.

a) Compare the second integral to an integral over the same interval of a simpler function to try to find a T so that the improper integral to be dropped will have a value smaller than 10^{-5}. *There is no unique correct answer to this part of the problem!*

b) Use a numerical technique to try to evaluate the first integral with an error of magnitude at most 10^{-5}. *You will need a programmable calculator or a computer for this part!**

c) Compute $\displaystyle\int_1^\infty \frac{e^{-x^3}}{1+x^2}\,dx$ with an error of magnitude at most $2\cdot 10^{-5}$.

4. Consider the following sequences:

a) $a_n = \left(1+\left(\frac{1}{n}\right)\right)^n$ b) $b_n = \left(1+\left(\frac{1}{n^2}\right)\right)^n$ c) $c_n = \left(1+\left(\frac{1}{\sqrt{n}}\right)\right)^n$

Find the first five terms of each of these sequences. What can you say about the convergence of each of these sequences? (Why?)

5. a) If you enter 5 in your 10 digit calculator, and then hit the square root button 20 times, you'll probably get:

$$1.00000\,15348$$

and if you then subtract 1 and multiply by 1,048,576 you'll get:

$$1.60943\,91475$$

but on the other hand, the same calculator will tell you that $\ln 5$ is:

$$1.60943\,79124$$

and I wonder, is this just a coincidence? **Well, is it?** Hint: 1,048,576 is 2^{20}.

b) Given a positive number, x, outline a strategy for computing $\ln x$ only with the "primitive" arithmetic operations ($+$, $\times$, $-$, $/$) and square root ($\sqrt{}$). Your strategy should involve asserting (and verifying) that a certain sequence which can be easily computed with the listed operations always converges to $\ln x$.

6. a) Graph the functions $f(x) = \arctan x$ and $g(x) = \dfrac{\pi}{2}\tanh\left(\dfrac{2x}{\pi}\right)$. What is the slope at the origin of each curve?

b) Evaluate $\displaystyle\lim_{x\to\infty} x\left(\frac{\pi}{2} - f(x)\right)$ and $\displaystyle\lim_{x\to\infty} e^{4x/\pi}\left(\frac{\pi}{2} - g(x)\right)$.

c) Suppose that you were shown superimposed graphs of these two curves and asked to match each curve with the corresponding formula (compare problem 2 of set #1). You have left your calculator at home and so cannot compute any specific values of arctan or tanh. How would b) enable you to make this distinction?

Comment Section 3.9 of the text has basic information about hyperbolic functions.

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* `Maple` has an implenmentation of of the Trapezoid Rule. Try `help(trapezoid);` and also `with (Student[CalculusI]);`.