Each student should do **one** problem and hand it in on Tuesday, November 22 (yes, the exam day). You can discuss the problems with each other or with me (I encourage you to do so because these problems are challenging!) but your writeups should be your own.

Solutions will be graded both for mathematical content and for exposition, which should be in **complete English sentences**. You don't need to show every computational detail, but please give enough information to allow readers to follow your reasoning. You may include appropriate diagrams, which should be carefully drawn and labeled.

1. **Fourier series**

a) Suppose $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ converges for all x. Explain why $f(x+2\pi) = f(x)$ for all x.

b) Here suppose that $a_n = \frac{1}{2^n}$ so that $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\sin(nx)$. Explain why the series converges for all x. Find a positive integer N so that the partial sum $S_N(x) = \sum_{n=1}^{N} \frac{1}{2^n}\sin(nx)$ is always within .01 of $f(x)$. That is, find N so that $|f(x) - S_N(x)| < .01$ for all x. Submit an accurate graph of this $S_N(x)$ on the interval $[0, 2\pi]$, preferably drawn by `Maple` or some other good graphing program.

c) Here suppose that $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin(n!x)$ (read the formula carefully!). Explain why the series converges for all x. Find a positive integer N so that the partial sum $S_N(x) = \sum_{n=1}^{N} \frac{1}{n^2}\sin(n!x)$ is always within .01 of $f(x)$. That is, find N so that $|f(x) - S_N(x)| < .01$ for all x. Submit an accurate graph of this $S_N(x)$, preferably drawn by `Maple` or some other good graphing program.

2. **A computer science problem?**

The following problem is quoted directly from D. Knuth, *The Art of Computer Programming*, the standard (multivolume) reference for theoretical computer science. This problem is in volume 1, *Fundamental Algorithms*.

If $f(x) = \sum_{k\geq 0} a_k x^k$, and this series converges for $x = x_0$, then show that

$$\sum_{k\geq 0} a_k x_0^k H_k = \int_0^1 \frac{f(x_0) - f(x_0 y)}{1-y}\,dy.$$

Here the numbers H_k are defined to be the partial sums of the harmonic series: $H_0 = 0$; $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ for $k \geq 1$.

Hint Expand everything on the right side in a power series in x_0. What is the coefficient of x_0^k? You should also remember the formula for the sum of a *finite* geometric series.

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3. **A silly formula**

The problem below is frequently attributed to Atle Selberg, a permanent member in mathematics of the Institute for Advanced Study, with the remark that he discovered it while a teenager. This is a nice legend, but the problem apparently also appears in an English calculus book of the 1920's and perhaps before. So: verify that

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}.$$

Also find the value of the sum to an accuracy of $\pm .001$ (with an explanation).

Comment I don't know any use for this formula besides its lovely existence.

Hint Write everything as e^{STUFF}, use exponential's Taylor series, and then a reduction formula for $\int x^n (\ln x)^m \, dx$ used improperly.

4. **An old problem**

For which α does

$$\int_0^{\infty} \frac{x^\alpha}{1+x^\alpha} dx$$

converge? (This is a problem from a text by G. H. Hardy called *Pure Mathematics*, which was the principle "calculus" source for several generations of English mathematicians and scientists.)

5. **Integral computations by physicists**

One of the physicists' favorite methods of computing definite integrals is illustrated by the examples in this problem. The trick is *very* slick. The general idea is: put in a parameter, differentiate with respect to that parameter, and see what happens.

a) Suppose $\mathcal{F}(a) = \displaystyle\int_0^{\infty} e^{-ax}\,dx$. Compute $\mathcal{F}(a)$ directly. Then differentiate both sides N times with respect to a, and set $a = 1$. The result should have some resemblance to a formula we used in the development of the Gamma function.

b) Suppose $\mathcal{H}(a) = \displaystyle\int_0^{\infty} \frac{1}{x^2+a}\,dx$. Compute $\mathcal{H}(a)$ directly. Then differentiate both sides N times with respect to a. The result, after a little bit of algebra, is a formula for $\displaystyle\int_0^{\infty} \frac{1}{(x^2+a)^N}\,dx$, where a is a positive number and N is a positive integer. A direct computation would be *horrible**.

* Of course, the mathematician hastens to say, "The justification for this method is not obvious." The physicist replies, "But it works."