Error analysis of the Trapezoid Rule

Dedicated to the 192 students who helped and kept me honest during class on Monday, October 3, 2005

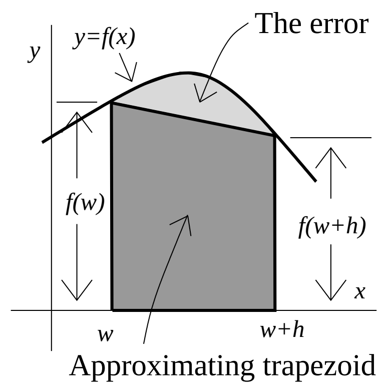
Two integrations by parts

We look closely at one "panel" of the trapezoid rule. The goal is to compare the true value of the integral from w to $w+h$ with the Trapezoid Rule approximation. Here w is x_j in the original setup, and $h = \frac{b-a}{n}$, so $w + h = x_{j+1}$.

$$\int_w^{w+h} f(x)\,dx = (x+C)f'(x)\Big]_{x=w}^{x=w+h} - \int_w^{w+h} (x+C)f'(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$\left.\begin{aligned} u &= f(x) \\ dv &= dx \end{aligned}\right\} \left\{\begin{aligned} du &= f'(x)\,dx \\ v &= x + C \end{aligned}\right.$$



The unusual (to me) choice of v above is used to make the *boundary term* $(x+C)f(x)\Big]_{x=w}^{x=w+h}$ equal to the trapezoid rule approximation. The trapezoid has bases $f(w)$ and $f(w+h)$, and height equal to h. So its area is the average of the bases multiplied by the height: $\frac{1}{2}\left(f(w)+f(w+h)\right)h$. We'll choose C so that $(x+C)f(x)\Big]_{x=w}^{x=w+h} = (w+h+C)f(w+h) - (w+C)f(w)$. If $C = -w - \frac{h}{2}$ then these are equal. Why? Well, $(w+h+C)f(w+h) - (w+C)f(w)$ becomes $(w+h+[-w-\frac{h}{2}])f(w+h) - (w+[-w-\frac{h}{2}])f(w) = \frac{h}{2}f(w+h) - \left(-\frac{h}{2}\right)f(w)$. Magically the minus signs cancel, and things work out.

We will call $\int_w^{w+h} f(x)\,dx$ the ***True Value*** and we will call $\frac{1}{2}\left(f(w)+f(w+h)\right)h$ the Trapezoid approximation, abbreviated ***Trap. approx.*** Then

$$\textbf{True Value} = \textbf{Trap. approx.} - \int_w^{w+h} (x+C)f'(x)\,dx$$

So the error is that integral, which will be analyzed using integration by parts. Forget the minus sign temporarily, and do this:

$$\int_w^{w+h} (x+C)f'(x)\,dx = \left(\tfrac{1}{2}(x+C)^2 + D\right) f'(x)\Big]_{x=w}^{x=w+h} - \int_w^{w+h} \left(\tfrac{1}{2}(x+C)^2 + D\right) f''(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

$$\left.\begin{aligned} u &= f'(x) \\ dv &= (x+C)\,dx \end{aligned}\right\} \left\{\begin{aligned} du &= f''(x)\,dx \\ v &= \tfrac{1}{2}(x+C)^2 + D \end{aligned}\right.$$

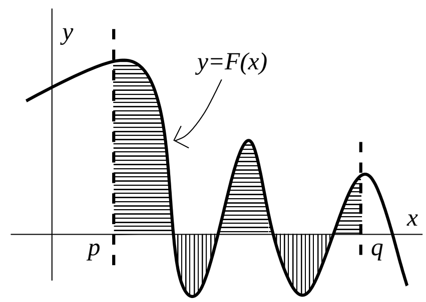
Again, to me, $\frac{1}{2}(x+C)^2+D$ is an even more ludicrous choice of v than the first one. Well, remember that we had $C=-w-\frac{h}{2}$, so that $\left(\frac{1}{2}(x+C)^2+D\right)f'(x)\Big]_{x=w}^{x=w+h} = \left(\frac{1}{2}\left(w+h-w-\frac{h}{2}\right)^2+D\right)f(w+h)-\left(\frac{1}{2}\left(w-w-\frac{h}{2}\right)^2+D\right)f(w)$. Stay calm, very calm, and do a bit more: this result is $\left(\frac{1}{8}h^2+D\right)f'(w+h)-\left(\frac{1}{8}(-h)^2+D\right)f'(w)$. If we choose D to be $-\frac{1}{8}h^2$ this boundary term is zero: it drops out totally! Now we have (minus signs cancel):

$$\textbf{\textit{True Value}} = \textbf{\textit{Trap. approx.}} + \int_w^{w+h} \left(\tfrac{1}{2}(x+C)^2+D\right)f''(x)\,dx$$

with the agreed values of C and D. Therefore $\int_w^{w+h}\left(\frac{1}{2}(x+C)^2+D\right)f''(x)\,dx$ must be the ***Error***, and we will work on it.

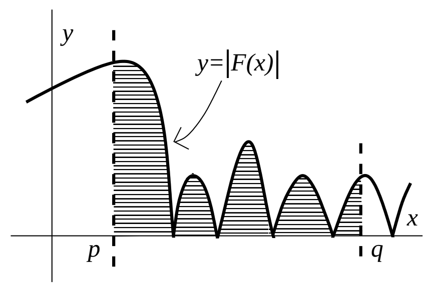
Estimating the *Error*

In practice people are usually interested in the magnitude or absolute value of the error. Consider how big $\left|\int_p^q F(x)\,dx\right|$ can be. Look at the picture. The regions above the x-axis give positive "weight" to the integral. They are drawn with horizontal marking. The regions below the axis are negatively weighted. They are shown with vertical marking, and might cancel some of the positive weight.



In general it can be *very* difficult to analyze the size of the result We want something very simple. ⇒ **If it isn't simple, people won't use it!** ⇐ So here we are focused on accuracy and usability.

Suppose we ignore the cancellations. That is, look at $y=|F(x)|$ instead of $y=F(x)$. Certainly we give up something. The cancellations may make the integral smaller. But, as in our case, it may be easier to understand the size of the integral $\int_p^q |F(x)|\,dx$ than the other integral. So we will use the inequality $\left|\int_p^q F(x)\,dx\right| \le \int_p^q |F(x)|\,dx$.



The integral we want to look at is therefore $\int_w^{w+h}\left|\left(\frac{1}{2}(x+C)^2+D\right)f''(x)\right|\,dx = \int_w^{w+h}\left|\frac{1}{2}(x+C)^2+D\right||f''(x)|\,dx$. Suppose we know *some convenient overestimate for the values of* $f''(x)$, which we will call K. Then $\int_w^{w+h}\left|\frac{1}{2}(x+C)^2+D\right||f''(x)|\,dx \le K\int_w^{w+h}\left|\frac{1}{2}(x+C)^2+D\right|\,dx$.

We're getting closer to the end. What can we say about the function $\left|\frac{1}{2}(x+C)^2+D\right|$ on the interval $[w,w+h]$? Remember that $C=-w-\frac{h}{2}$ and $D=-\frac{1}{8}h^2$. Then $\left|\frac{1}{2}(x+C)^2+D\right| = \left|\frac{1}{2}(x-w-\frac{h}{2})^2-\frac{1}{8}h^2\right|$. We will expand the square so $(x-w-\frac{h}{2})^2 = x^2+w^2+\frac{h^2}{4}-2xw-2x\left(\frac{h}{2}\right)+2wx\left(\frac{h}{2}\right)$. Therefore $\left|\frac{1}{2}\left(x-w-\frac{h}{2}\right)^2-\frac{1}{8}h^2\right| = \left|\frac{1}{2}\left(x^2+w^2+\frac{h^2}{4}-2xw-2x\left(\frac{h}{2}\right)+2w\left(\frac{h}{2}\right)\right)-\frac{1}{8}h^2\right| = \ldots$ And some "accidents" happen.

Again, please realize that just as in the preceding choices of parts in the antidifferentiation, these are accidents which have been planned. Keep this advice in mind as we go on. The remainder will "merely" be algebra.

$$\left|\frac{1}{2}\left(x^2+w^2+\frac{h^2}{4}-2xw-2x\left(\frac{h}{2}\right)+2w\left(\frac{h}{2}\right)\right)-\frac{1}{8}h^2\right| =$$
$$\left|\frac{1}{2}x^2+\frac{1}{2}w^2+\frac{h^2}{8}-xw-\frac{1}{2}xh+w\left(\frac{h}{2}\right)-\frac{1}{8}h^2\right| =$$
$$\left|\frac{1}{2}x^2+\frac{1}{2}w^2-xw-\frac{1}{2}xh+w\left(\frac{h}{2}\right)\right| = \left|\frac{1}{2}(w-x)^2+\frac{h}{2}(w-x)\right|$$

Now look even more closely. $\left|\frac{1}{2}(w-x)^2+\frac{h}{2}(w-x)\right| = \frac{1}{2}|w-x|\,|w-x+h|$.

Since x is in the interval $[w, w+h]$, $w-x$ must be *negative*, while $w-x+h$ must be *positive*. Why? Look: $w<x<w+h$ implies $w-x<0<w+h-x=w-x+h$.

The product of the two terms will be negative, and the absolute value of a negative number is *minus* that number: $\left|\frac{1}{2}(w-x)^2+\frac{h}{2}(w-x)\right| = -\left(\frac{1}{2}(w-x)^2+\frac{h}{2}(w-x)\right)$. So:

$$\textbf{\textit{Error}} = \int_w^{w+h} \left|\tfrac{1}{2}(x+C)^2+D\right|\,dx = \int_w^{w+h} -\left(\tfrac{1}{2}(w-x)^2+\tfrac{h}{2}(w-x)\right)dx$$

I'll compute this with care. The result is $-\left(-\frac{1}{6}(w-x)^3-\frac{h}{4}(w-x)^2\right)\Big]_{x=w}^{x=w+h}$. The terms with $x=w$ are 0. When $x=w+h$ we get $-\left(-\frac{1}{6}(-h)^3-\frac{h}{4}(-h)^2\right)$. The final result is $\frac{h^3}{12}$.

The summary

Why is this used? Well, let's see: now we know the error in $[w,w+h]$ is at most $K\frac{h^3}{12}$. Here K is some convenient overestimate of f'', and $h=\frac{b-a}{n}$. There are n panels, so the total error could be as much as $n\cdot K\frac{h^3}{12} = nK\frac{\left(\frac{b-a}{n}\right)^3}{12} = K\frac{(b-a)^3}{12n^2}$.

This is where the textbook should take over ...

10/4/2005