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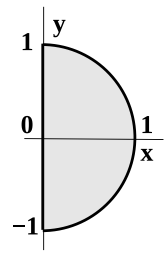
640:251:05–10 **Answers to the Second Exam** **4/9/2006**

(8) 1. If $z = 5y + f(3x^2 - 7y^2)$, where f is differentiable, show that $7y\frac{\partial z}{\partial x} + 3x\frac{\partial z}{\partial y} = 15x$.

Answer $z_x = f'(3x^2-7y^2)(6x)$ and $z_y = 5+f'(3x^2-7y^2)(-14y)$ so that $7yz_x+3xz_y = 7y\big(f'(3x^2-7y^2)(6x)\big)+3x\big(5+f'(3x^2-7y^2)(-14y)\big) = (42xy-42xy)\,f'(3x^2-7y^2)+15x = 15x$.

(12) 2. The average value of a function f defined in a region R of $\mathbb{R}^2$ is $\dfrac{\iint_R f\,dA}{\iint_R dA}$. Compute the average distance to the origin of points in the right half plane ($x \geq 0$) which are inside the unit circle.

Answer The distance to the origin of a point (x, y) is $\sqrt{x^2+y^2}$. The region R is shown to the right. Polar coordinates are appropriate for the integral $\iint_R f\,dA$. Then the function becomes r and dA is $r\,dr\,d\theta$ and the limits on the r integral are 0 and 1, and on the θ integral, the limits are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Therefore $\iint_R f\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^1 r^2\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}r^3\Big]_{r=0}^{r=1} d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\,d\theta = \frac{\pi}{3}$. Half of the area inside the unit circle is $\frac{\pi}{2}$ so that the average value is $\frac{2}{3}$.



Comment Using `Maple`'s random function, I got 0.66698 as the average value with 10^5 trials.

(12) 3. Leonhard Euler (1707–1783) was a great and very prolific mathematician. He published *Institutiones Calculi Differentialis* (*Methods of the Differential Calculus*) in 1755. It was an influential text, and was the first source of criteria for discovering local extrema of functions of several variables. In it Euler investigated the following specific example: $V = x^3 + y^2 - 3xy + \frac{3}{2}x$. He asserted that V has a minimum both at $\left(1, \frac{3}{2}\right)$ and at $\left(\frac{1}{2}, \frac{3}{4}\right)$. Was Euler correct?

Reference: *A History of Mathematics* by Victor J. Katz, Harper Collins, 1993, p. 517.

You must check that Euler found all the critical points and only the critical points, and also that he classified them correctly.

Answer $V_x = 3x^2 - 3y + \frac{3}{2}$ and $V_y = 2y - 3x$. Critical points occur where both V_x and V_y are 0. There $2y = 3x$ or $y = \frac{3}{2}x$, so the V_x condition becomes: $3x^2 - \frac{9}{2}x + \frac{3}{2} = 0$ or $6x^2 - 9x + 3 = 0$ which factors* into $(2x-1)(3x-3) = 0$. The critical points *are* as Euler wrote: $\left(1, \frac{3}{2}\right)$ and $\left(\frac{1}{2}, \frac{3}{4}\right)$. Now test the type of the critical points: $V_{xx} = 6x$, $V_{xy} = -3$, $V_{yx} = -3$, $V_{yy} = 2$. So the Hessian is $\det\begin{pmatrix} 6x & -3 \\ -3 & 2 \end{pmatrix} = 12x - 9$. At $\left(1, \frac{3}{2}\right)$, the Hessian is $12 - 9 > 0$, and $V_{xx} = 6 > 0$, so this critical point is a local minimum. At $\left(\frac{1}{2}, \frac{3}{4}\right)$, the Hessian is $12 \cdot \frac{1}{2} - 9 = -3 < 0$, which makes this critical point a saddle point. Euler was wrong!

(12) 4. Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint: $f(x, y, z) = 8x - 4z;\ x^2 + 6y^2 + 3z^2 = 5$.

Answer Suppose $g(x, y, z) = x^2 + 6y^2 + 3z^2$. The vector equation $\nabla f = \lambda\nabla g$ turns into $\begin{cases} 8 = \lambda(2x) \\ 0 = \lambda(12y) \\ -4 = \lambda(6z) \end{cases}$.

The second equation shows that either λ or y must be 0 but if $\lambda = 0$ then there are no solutions to the first or third equations. So $y = 0$ and $\lambda \neq 0$. The first and third equations lead to $x = \frac{4}{\lambda}$ and $z = -\frac{2}{3\lambda}$. The constraint equation $g(x, y, z) = 5$ then becomes $\left(\frac{4}{\lambda}\right)^2 + 0^2 + 3\left(-\frac{2}{3\lambda}\right)^2 = 5$ so that $(16 + \frac{4}{3} = 5\lambda^2$ so that $\lambda = \pm\sqrt{\frac{52}{15}}$. The values for x and z follow: $x = \pm 4\sqrt{\frac{15}{52}}$ AND $z = \mp\frac{2}{3}\sqrt{\frac{15}{52}}$ (the signs are linked). And $f\left(\pm 4\sqrt{\frac{15}{52}}, 0, \mp\frac{2}{3}\sqrt{\frac{15}{52}}\right) = \pm 32\sqrt{\frac{15}{52}} - \left(\mp\frac{8}{3}\sqrt{\frac{15}{52}}\right) = \pm\frac{104}{3}\sqrt{\frac{15}{52}}$. The maximum value is $\frac{104}{3}\sqrt{\frac{15}{52}}$ and the minimum value is $-\frac{104}{3}\sqrt{\frac{15}{52}}$.

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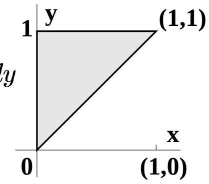
* Even then textbook problems were predictable.

(10) 5. Compute $\iint_R e^{3x/y}\,dA$ where the region R is the triangle shown.

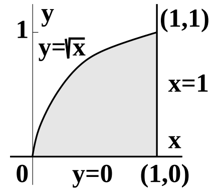


Answer $\iint_R e^{3x/y}\,dA = \int_{y=0}^{y=1}\int_{x=0}^{x=y} e^{3x/y}\,dx\,dy = \int_{y=0}^{y=1} \frac{y}{3}e^{3x/y}\Big]_{x=0}^{x=y} dy = \int_{y=0}^{y=1}\left(\frac{e^3}{3}y - \frac{1}{3}y\right)dy$ $= \frac{1}{6}e^3 - \frac{1}{6}$.

Comment The iterated integral in $dy\,dx$ order leads to an impossible antiderivative.

(16) 6. Evaluate the triple integral $\iiint_E 7xy\,dV$ where E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$.

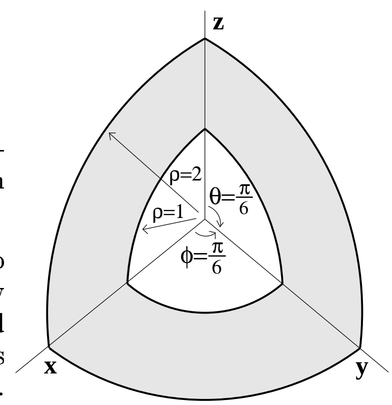
Answer I'll put dz the most inside, because then the limits on z then will be from $z = 0$ on the xy-plane to $z = 1 + x + y$. The base of the solid is shown to the right and maybe $dy\,dx$ is easiest. The triple integral becomes a triply iterated integral: $\iiint_E 7xy\ dV = \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{x}}\int_{z=0}^{z=1+x+y} 7xy\,dz\,dy\,dx$.
We evaluate: $\int_{z=0}^{z=1+x+y} 7xy\,dz = 7xyz\Big]_{z=0}^{z=1+x+y} = 7xy(1 + x + y) = 7xy + 7x^2y + 7xy^2$,
and then $\int_{y=0}^{y=\sqrt{x}} 7xy+7x^2y+7xy^2\,dy = \frac{7}{2}xy^2 + \frac{7}{2}x^2y^2 + \frac{7}{3}xy^3\Big]_{y=0}^{y=\sqrt{x}} = \frac{7}{2}x^2+\frac{7}{2}x^3+\frac{7}{3}x^{5/2}$,
and finally $\int_{x=0}^{x=1} \frac{7}{2}x^2 + \frac{7}{2}x^3 + \frac{7}{3}x^{5/2}\,dx = \frac{7}{6}x^3 + \frac{7}{8}x^4 + \frac{2}{3}x^{7/2}\Big]_{x=0}^{x=1} = \frac{7}{6} + \frac{7}{8} + \frac{2}{3} = \frac{65}{24}$.



(12) 7. Consider the iterated triple integral $\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.
a) Compute the integral.

Answer First, $\int_0^{\pi/2}\rho^2\sin\phi\,d\rho = \frac{1}{3}\rho^3\sin\phi\Big]_{\rho=1}^{\rho=2} = \frac{7}{3}\sin\phi$ so $\int_0^{\pi/2}\frac{7}{3}\sin\phi\,d\phi$ $= -\frac{7}{3}\cos\phi\Big]_{\phi=0}^{\phi=\pi/2} = \frac{7}{3}$; $\int_0^{\pi/2}\frac{7}{3}\,d\theta = \frac{7\pi}{6}$.
b) The integral evaluates the volume of a solid in $\mathbb{R}^3$ with spherical coordinates. Use complete English sentences possibly accompanied by a sketch to describe this solid.

Answer The bounds on the angles θ and ϕ restrict points in the solid to the first octant, where $x \geq 0$, $y \geq 0$, and $z \geq 0$. The bounds on ρ show that the distance to the origin of the solid's points must be between 1 and 2. Therefore the solid consists of that part of the region between spheres with center the origin and radius 1 and radius 2 which is in the first octant.



(18) 8. a) Compute $\int_C (x^2 + y)\,dx + (-x + 3)\,dy$ if C is the curve $y = x^3$ from $x = 0$ to $x = 2$.

Answer Parameterize C with $\begin{cases} x(t) = t \\ y(t) = t^3 \end{cases}$ from $0 \leq t \leq 2$. Then $\begin{cases} dx = dt \\ dy = 3t^2\,dt \end{cases}$ so $\int_C (x^2 + y)\,dx + (-x + 3)\,dy = \int_0^2 \left((t^2 + t^3) + (-t + 3)(3t^2)\right)dt = \int_0^2 10t^2 - 2t^3\,dt = \frac{10}{3}t^3 - \frac{2}{4}t^4\Big]_0^2 = \frac{80}{3} - 8 = \frac{56}{3}$.

b) Suppose $\mathbf{V}$ is the vector field defined by $\mathbf{V}(x, y) = (x^2 + y)\mathbf{i} + (x + 3)\mathbf{j}$. Find a scalar function $f(x, y)$ defined everywhere in the plane so that $\nabla f = \mathbf{V}$. You must show a process leading from $\mathbf{V}$ to f. Compute $\int_C \mathbf{V}\cdot\mathbf{T}\,ds$ if C is the curve $\begin{cases} x(t) = 5\cos t - 2(\cos t)^4 \\ y(t) = 4 - 2(\sin t)^2 \end{cases}$ for t in the interval $\left[0, \frac{\pi}{2}\right]$. Here $\mathbf{T}$ is the unit tangent vector to the curve, C.

Answer If $\nabla f = \mathbf{V}$ then $\frac{\partial f}{\partial x} = x^2 + y$ and $\frac{\partial f}{\partial y} = x + 3$. Therefore $f(x, y) = \int (x^2 + y)\,dx = \frac{1}{3}x^3 + xy + C_1(y)$ and $f(x, y) = \int (x + 3)\,dy = xy + 3y + C_2(x)$. We match the two descriptions of f: $\frac{1}{3}x^3 + xy + C_1(y) = xy + 3y + C_2(x)$. So $\frac{1}{3}x^3$ is concealed in $C_2(x)$ and $3y$ is in $C_1(y)$. One potential is $f(x, y) = \frac{1}{3}x^3 + xy + 3y$. At the START of the curve C, when $t = 0$, $(x(0), y(0)) = (5 - 2, 4) = (3, 4)$. At the END, when $t = \frac{\pi}{2}$, $\left(x\left(\frac{\pi}{2}\right), \left(\frac{\pi}{2}\right),\right) = (0, 4-2) = (0, 2)$. Thus the line integral is equal to $f(0, 2) - f(3, 4) = 6 - (9+12+12) = -27$.