Formula sheet for the final exam in Math 251:05-10, spring 2006

The distance from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$ is $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$.

The distance from $P_1(x_1,y_1,z_1)$ to plane $ax+by+cz+d=0$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$.

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\mathbf{n}=\langle a,b,c\rangle$.

Line: $\begin{cases} x = x_0+at \\ y = y_0+bt \\ z = z_0+ct \end{cases}$ through (x_0,y_0,z_0) in direction $\langle a,b,c\rangle$.

$|\mathbf{a}|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$ if $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$.

$\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta$ (If $=0$, then $\mathbf{a}\perp\mathbf{b}$.) $\quad |\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$ (If $\mathbf{a}||\mathbf{b}$, this $=0$.)

$\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a} \quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} \quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\text{comp}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} \quad \text{proj}_{\mathbf{a}}\mathbf{b}=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$

Volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$

Arc length: $\int_a^b |\mathbf{r}'(t)|\,dt$; $\frac{ds}{dt}=|r'(t)|$; $\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$; $\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$; $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$

$$\kappa=\left|\frac{d\mathbf{T}}{ds}\right|=\frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}\overset{2\ \text{dim}}{=}\frac{|y''(t)x'(t)-x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^{3/2}}\overset{y=f(x)}{=}\frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

$\tau=\dfrac{(\mathbf{r}'(t)\times\mathbf{r}'(t))\cdot\mathbf{r}'''(t)}{|\mathbf{r}'(t)\times\mathbf{r}''(t)|^2}$. $\quad$ Frenet-Serret: $\dfrac{d\mathbf{T}}{ds}=\kappa\mathbf{N}$, $\dfrac{d\mathbf{N}}{ds}=-\kappa\mathbf{T}+\tau\mathbf{B}$, $\dfrac{d\mathbf{B}}{ds}=-\tau\mathbf{N}$.

Tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$: $z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

Linear approximation to $f(x,y)$ at (a,b): $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent plane to $F(x,y,z)=0$:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

If y implicitly defined by $y=f(x)$ in $F(x,y)=0$ then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$.

If z implicitly defined by $z=f(x,y)$ in $F(x,y,z)=0$ then $z_x=-\dfrac{F_x}{F_z}$ and $z_y=-\dfrac{F_y}{F_z}$.

$\nabla f=\langle f_x,f_y,f_z\rangle=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k} \quad D_{\mathbf{u}}f(x,y,z)=\nabla f(x,y,z)\cdot\mathbf{u}$

Some chain rules:

If $z=f(x,y)$ and $x=x(t)$ and $y=y(t)$, then $\dfrac{dz}{dt}=\dfrac{\partial f}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial f}{\partial z}\dfrac{dy}{dt}$.

If $z=f(x,y)$ and $x=g(s,t)$ and $y=h(s,t)$, then $\dfrac{\partial z}{\partial s}=\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$.

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$. Let $H=H(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

a) If $H>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $H>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $H<0$, then $f(a,b)$ is not a local maximum or minimum (f has a saddle point).

Lagrange multipliers for one constraint

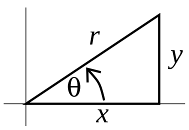
If G(the variables) = a constant is the constraint and we want to extremize the objective function, F(the variables), then the extreme values can be found among F's values of the solutions to the system of equations $\nabla G = \lambda \nabla F$ (a vector abbreviation for the equations $\lambda \frac{\partial F}{\partial \star} = \frac{\partial G}{\partial \star}$ where $\star$ is each of the variables) **and** the constraint equation.

Polar coordinates

$x = r\cos\theta \quad y = r\sin\theta$

$r^2 = x^2 + y^2 \quad \theta = \arctan\left(\frac{y}{x}\right)$

$dA = r\,dr\,d\theta$

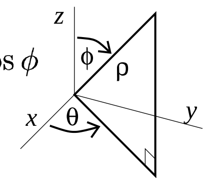


Spherical coordinates

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2$

$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$



Total mass of a mass distribution $\rho(x,y,z)$ over a region R of $\mathbb{R}^3$ is $\int\int\int_R \rho(x,y,z)\,dV$.

Line integral formulas

$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$

$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(x(t),y(t))x'(t)\,dt + Q(x(t),y(t))y'(t)\,dt$

Green's Theorem

$\int_C P\,dx + Q\,dy = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

These P, Q pairs will give R's area $\begin{cases} P{=}-y \text{ and } Q{=}0 \\ P{=}0 \text{ and } Q{=}x \\ P{=}-\frac{1}{2}y \text{ and } Q{=}\frac{1}{2}x \end{cases}$

A **conservative vector field** $\mathbf{V} = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a **gradient vector field**: there's $f(x,y)$ with $\nabla f = \mathbf{V}$ so $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$. f is a **potential** for $\mathbf{V}$. A conservative vector field is **path independent**. Work done by such a vector field over a **closed curve** is 0. For V conservative with potential f: $\int_C P\,dx + Q\,dy = f(\text{THE END}) - f(\text{THE START})$.

If $P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is conservative, then $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. If the region is **simply connected** (means **no holes**) then the converse is true, and f is both $\int P(x,y)\,dx$ and $\int Q(x,y)\,dy$.

Surfaces: If $\mathbf{n}$ is a choice of normal for S, **flux** is $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$.

Parametrically: $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$; $\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$ is $\perp$ to S; $dS = \left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right| dA_{uv}$.

As a graph: $z = f(x,y)$; $-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$ is $\perp$ to S; $dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\,dA_{xy}$.

If $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ and $\mathbf{F}$ is a vector field then $\begin{cases} \text{curl } F = \nabla \times \mathbf{F}, \text{ a vector field.} \\ \text{div } F = \nabla \cdot \mathbf{F}, \text{ a function.} \end{cases}$

Potentials in $\mathbb{R}^3$

If $\mathbf{F} = \nabla f$ and C is a curve, then $\int_C P\,dx + Q\,dy + R\,dz = f(\text{THE END}) - f(\text{THE START})$, path independence holds, the work over a closed curve is 0, and $\text{curl}(\nabla f) = 0$. Conversely, if F is defined in all of $\mathbb{R}^3$ with curl $F = 0$ (the cross-partials "match") then $\mathbf{F}$ has a potential, f, so $\nabla f = \mathbf{F}$. f is obtained by comparing partial integrals of the components of $\mathbf{F}$.

Stokes' Theorem (As you "walk" along C, S is to the left and $\mathbf{n}$ is up.)

$\left[\int\int_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS =\right] \quad \int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot d\mathbf{r} \quad \left[= \int_C P\,dx + Q\,dy + R\,dz\right]$

Divergence Theorem ($\mathbf{n}$ is unit *outward* normal to E, a region in $\mathbb{R}^3$ with boundary S.)

$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int\int_E \text{div } F\,dV \quad \left[= \int\int\int_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,dV\right]$