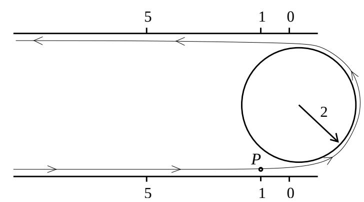
This is probably the most straightforward exam *ever* given by this lecturer in a third semester calculus course. The majority of the problems were exactly copied from textbook problems suggested for homework. Other problems were either similar to textbook problems or taken from review material.

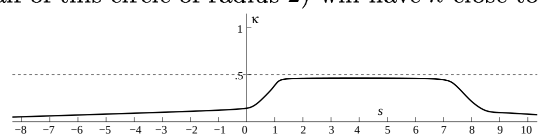
(8) 1. Suppose the position vector of a curve is given by $\mathbf{r}(t) = \langle \sin(2t), \cos(3t), t^2 \rangle$. Find the unit tangent vector to this curve when $t = \pi$. That is, find $\mathbf{T}(\pi)$.
Answer The velocity vector is $\langle 2\cos(2t), -3\sin(3t), 2t\rangle$ and when $t = \pi$ this is $\langle 2, 0, 2\pi\rangle$. $\mathbf{T}(\pi)$ is this vector normalized, so $\mathbf{T}(\pi) = \frac{\langle 2,0,2\pi\rangle}{\sqrt{2^2+(2\pi)^2}} = \left\langle \frac{1}{\sqrt{1+\pi^2}}, 0, \frac{\pi}{\sqrt{1+\pi^2}} \right\rangle$.

(8) 2. Find a vector parameterization for the line which passes through $(1, 1, 1)$ and $(3, -5, 2)$. 12.2 #31
Answer A vector from $(1, 1, 1)$ to $(3, -5, 2)$ is $\langle 2, -6, 1\rangle$ and therefore one answer for the vector parameterization is $\langle 1 + 2t, 1 - 6t, 1 + t\rangle$.

(8) 3. A point moves along the curve drawn **[at right]** in the direction indicated. Its motion is parameterized by arc length, s, so that it is moving at unit speed. Arc length is measured from the point P (both backward and forward). The curve is intended to continue indefinitely both forward and backward in s, with its forward motion coming closer and closer to the indicated straight line, and backward, coming closer and closer to the other indicated straight line. The numbers on each line indicate distance along that line.



Answer Certainly $\lim_{s\to+\infty} \kappa(s) = 0$ and $\lim_{s\to-\infty} \kappa(s) = 0$ since the curve in both directions is getting flatter and flatter, more like a straight line which has $\kappa = 0$. The part of the curve which is close to the circle (roughly from $s = 1$ to $s = 1 + 2\pi$, since 2π is the length of half of this circle of radius 2) will have κ close to .5, the curvature of a circle of radius 2. The curvature looks like it increases smoothly from close to 0 to close to .5, stays there for about 2π, and then decreases smoothly to close to 0. Therefore the graph of κ appears as shown.



(8) 4. Find two vectors (which are not multiples of each other) that are both orthogonal to $\langle 2, 0, -3\rangle$. 12.3 #31
Answer Label $\langle 2, 0, -3\rangle$ as $\mathbf{v}$. Then $\mathbf{w}_1 = \langle 0, 1, 0\rangle$ is orthogonal to $\mathbf{v}$ because $\mathbf{v}\cdot\mathbf{w}_1 = 2\cdot 0+0\cdot 1+(-3)\cdot 0 = 0$. Also $\mathbf{w}_2 = \langle 3, 0, 2\rangle$ is orthogonal to $\mathbf{v}$ because $\mathbf{v} \cdot \mathbf{w}_2 = 2 \cdot 3 + 0 \cdot 1 + (-3) \cdot 2 = 6 - 6 = 0$. $\mathbf{w}_1$ and $\mathbf{w}_2$ are not multiples of each other because of the pattern of 0's. So if $a\mathbf{w}_1 = \langle 0, a, 0\rangle = \mathbf{w}_2 = \langle 3, 0, 2\rangle$ then (first component) $0 = 3$ which is false. Similarly, if $a\mathbf{w}_2 = \langle 3a, 0, 2a\rangle = \mathbf{w}_1 = \langle 0, 1, 0\rangle$ then (second component) $0 = 1$ which is false.

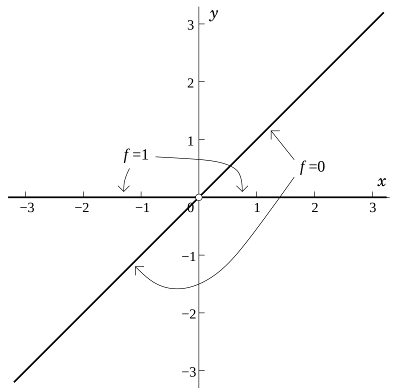
(8) 5. Calculate the cross product: $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times (\mathbf{i} + \mathbf{j} - 7\mathbf{k})$. 12.4 #20
Answer The cross product is $\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 4 \\ 1 & 1 & -7 \end{pmatrix} = (21 - 4)\mathbf{i} - (-14 - 4)\mathbf{j} + (2 - (-3))\mathbf{k} = 17\mathbf{i} + 18\mathbf{j} + 5\mathbf{k}$.

(8) 6. A bee with velocity vector $\mathbf{r}'(t)$ starts out at the origin at $t = 0$ and flies around for T seconds. Where is the bee located at time T if $\int_0^T \mathbf{r}'(u)\,du = 0$? What does the quantity $\int_0^T \|\mathbf{r}'(u)\|\,du$ represent? 13.3 #13
Answer Apply the Fundamental Theorem of Calculus to the components of the position and velocity vectors and then deduce that $\int_0^T \mathbf{r}'(u)\,du = \mathbf{r}(u)\Big]_0^T = \mathbf{r}(T) - \mathbf{r}(0)$. We know $\mathbf{r}(0) = 0$ (the phrase "starts out at the origin at $t = 0$") and since we are told that the integral is 0 also, we know $\mathbf{r}(T) = 0$. This means that at time T the bee is again at the origin.
$\mathbf{r}'(u)$ is the bee's velocity at time u, and $\|\mathbf{r}'(u)\|$ is the bee's speed at time u. The definite integral of the speed with respect to u from 0 to T is the total distance that the bee travels during that period (distance=rate·time).

Apis mellifera, Honey Bee

(10) 7. Sketch two level curves of $f(x,y)=\frac{x-y}{x+y}$ and *label* these curves appropriately.
Explain why $\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$ does or does not exist. Review question A

Answer When $f=0$ then $x-y=0$ so $x=y$, and this line (excluding $(0,0)$ to be precise since that point is not in the domain of f!) is a level curve. When $f=1$, then $\frac{x-y}{x+y}=1$ so $x-y=x+y$ and then the level curve is $y=0$, again excluding $(0,0)$. Many other level curves can be drawn.
The indicated limit does **NOT** exist. This is because as $(x,y)\to(0,0)$ along the line $y=x$, the value of f is 0, and as $(x,y)\to(0,0)$ along the line $y=0$, the value of f is 1. Since $1\neq 0$, the limit, which should be a unique number, does not exist.



(10) 8. If $h(x,z)=e^{2xz-x^2z^3}$, compute $\frac{\partial^2 h}{\partial x\partial z}$ when $x=2$ and $z=1$. Like 14.3 #43

Answer $h_x=e^{2xz-x^2z^3}(2z-2xz^3)$ so (product rule!) $h_{xz}=e^{2xz-x^2z^3}(2x-3x^2z^2)(2z-2xz^3)+e^{2xz-x^2z^3}(2-6xz^2)$. When $x=2$ and $z=1$ (notice that $2xz-x^2z^3=2\cdot 2-2^2=0$) this is $e^0(4-12)(2-4)+e^0(2-12)=-8(-2)+(-10)=6$.

(8) 9. Suppose $g(w)$ is any differentiable function of one variable. If $f(x,y)=g(x^2-3y)$, verify that $3\frac{\partial f}{\partial x}+2x\frac{\partial f}{\partial y}=0$. Part of problem 7 on an old exam

Answer $f_x=g'(x^2-3y)(2x)$; $f_y=g'(x^2-3y)(-3)$. Therefore $3\frac{\partial f}{\partial x}+2x\frac{\partial f}{\partial y}=3g'(x^2-3y)(2x)+(2x)g'(x^2-3y)(-3)=(6x-6x)g'(x^2-3y)=0$.

(12) 10. a) Find an equation of the tangent plane to the surface at the given point: $xz+2x^2y+y^2z^3=11$, $P=(2,1,1)$. 14.5 #42

Answer If $f(x,y,z)=xz+2x^2y+y^2z^3$, $\nabla f=\langle z+4xy, 2x^2+2yz^3, x+3y^2z^2\rangle$ and at P, the gradient is $\langle 9,10,5\rangle$, a vector perpendicular to the level surface. An equation for the plane tangent to the surface is $9(x-2)+10(y-1)+5(z-1)=0$.
b) If $f(x,y,z)=xz+2x^2y+y^2z^3$, find a unit vector in the direction of the largest directional derivative of f at $P=(2,1,1)$. What is the value of that directional derivative? (Information obtained in a) would allow you to answer these questions immediately.)

Answer A unit vector in the direction of the largest directional derivative is the gradient normalized, so this unit vector must be $\frac{\langle 9,10,5\rangle}{\sqrt{9^2+10^2+5^2}}=\left\langle\frac{9}{\sqrt{206}},\frac{10}{\sqrt{206}},\frac{5}{\sqrt{206}}\right\rangle$. The magnitude of that directional derivative is the magnitude of the gradient, and this is $\sqrt{206}$.

(12) 11. Locate and identify (max, min, saddle) all the critical points of $f(x,y)=\frac{x^3}{3}+\frac{y^2}{2}-xy$.

Comment To the right is a picture of part of the graph of $z=f(x,y)$ which contains the critical points. This may help, but your solution *must* be algebraic.

Answer $f_x=x^2-y$ and $f_y=y-x$ so at a c.p. $y=x$ and $y^2-y=0$. This means $(y^2-y=y(y-1))$ the c.p.'s are $(0,0)$ and $(1,1)$. The discriminant, D is $f_{xx}f_{yy}-(f_{xy})^2=(2x)(1)-(-1)^2=2x-1$. At $(0,0)$, D is -1, so this c.p. is a saddle point. At $(1,1)$, D is 1 and $f_{xx}=1$ (hey, $f_{yy}=1>0$ also!) so $(1,1)$ is a local minimum. This is consistent with the graph shown.