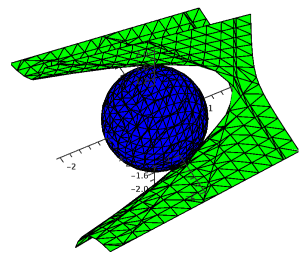
640:251:22–24 **Answers to the Second Exam** **11/16/2008**

Here are detailed answers to version A. Brief answers to version B are at the end.

(12) 1. Theoretical results imply that $x+3yz$ has a maximum and a minimum on the sphere $x^2+y^2+z^2=1$. Use Lagrange multipliers to find these maximum and minimum values.

Answer Suppose $f(x,y,z)=x+3yz$ and $g(x,y,z)=x^2+y^2+z^2$. Then $\nabla f=\langle 1,3z,3y\rangle$ and $\nabla g=\langle 2x,2y,2z\rangle$ so that the Lagrange multiplier equations are (including the constraint equation) $1=\lambda(2x)$, $3z=\lambda(2y)$, $3y=\lambda(2z)$, and $1=x^2+y^2+z^2$. Now we solve these equations. The first equation immediately tells us that neither x nor λ can be 0. The second and third equations imply if $y=0$ then $z=0$ (and *vice versa*). If both y and z are 0 then $1=x^2+y^2+z^2$ shows that $x=\pm1$ so the objective function $x+3yz$ is ±1. Can we do better (get larger and smaller values of the objective function)? If no variable is 0, the second and third equations can be rewritten as $\frac{3z}{2y}=\lambda$ and $\frac{3y}{2z}=\lambda$ so $\frac{3z}{2y}=\frac{3y}{2z}$ and $z^2=y^2$. Then λ must be $\pm\frac{3}{2}$ itself since $y=\pm z$. So $1=\lambda(2x)$ implies that $x=\pm\frac{1}{3}$ and $1=x^2+y^2+z^2$ gives $1=\left(\frac{1}{3}\right)^2+2z^2$ and $z=\pm\sqrt{\frac{4}{9}}$ and $y=\pm\sqrt{\frac{4}{9}}$. If $\lambda>0$, then $x>0$, and y and z have the same sign. If $\lambda<0$, then $x<0$, and y and z have opposite signs. With all $+$ signs, $x+3yz$ becomes $\frac{1}{3}+3\left(\frac{4}{9}\right)=\frac{5}{3}$. With $-$ signs for x and y and $+$ for z we get $-\frac{5}{3}$. These are the actual minimum and maximum values. To the right is a picture of the ball together with the surface $x+3yz=\frac{5}{3}$. They do indeed seem to be tangent (and at two points, the other corresponding to minus signs on y and z) just as the Lagrange multiplier method suggests.



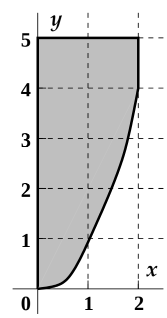
(12) 2. Suppose $I=\int_0^2\int_{x^2}^5 xy\,dy\,dx$.

a) Compute I.

Answer $\int_0^2\int_{x^2}^5 xy\,dy\,dx=\int_0^2 \frac{xy^2}{2}\Big|_{y=x^2}^{y=5}dx=\int_0^2 \frac{25}{2}x-\frac{x^5}{2}\,dx=\frac{25}{4}x^2-\frac{x^6}{12}\Big|_{x=0}^{x=2}=25-\frac{2^6}{12}=25-\frac{16}{3}=\frac{59}{3}$.

b) Use the axes to the right to sketch the region of integration for I.

Answer Shown to the right.

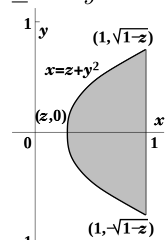


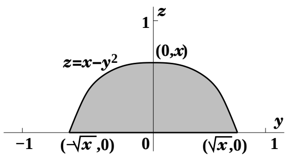
c) Write I as a sum of one or more $dx\,dy$ integrals. You do not need to compute the result!

Answer $\int_4^5\int_0^2 xy\,dx\,dy+\int_0^4\int_0^{\sqrt{y}} xy\,dx\,dy$.

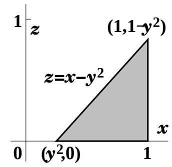
(12) 3. The coordinates (x,y,z) of points in a solid object A in $\mathbb{R}^3$ satisfy the inequalities $0\le z\le x-y^2$ and $0\le x\le 1$. Compute the triple integral of 1 over the object A. (This is the volume of A.) **Note** Four views of the object were given.

Answer A picture of a slice with z fixed is to the right, where $0<z<1$. The volume is $\int_0^1\int_z^1\int_{-\sqrt{x-z}}^{\sqrt{x-z}}1\,dy\,dx\,dz=\int_0^1\int_z^1 y\Big|_{y=-\sqrt{x-z}}^{y=\sqrt{x-z}}dx\,dz=\int_0^1\int_z^1 2\sqrt{x-z}\,dx\,dz=$
$\int_0^1 \frac{4}{3}(x-z)^{3/2}\Big|_{x=z}^{x=1}dz=\int_0^1\frac{4}{3}(1-z)^{3/2}dz=-\frac{8}{15}(1-z)^{5/2}\Big|_{z=0}^{z=1}=\frac{8}{15}$.





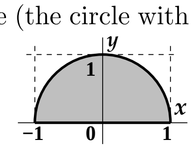
A picture of a slice with x fixed is to the left, where $0<x<1$ so another answer is $\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{x-y^2}1\,dz\,dy\,dx$.



$\int_{-1}^1\int_0^{1-y^2}\int_{z+y^2}^1 1\,dx\,dz\,dy$ is another answer. A slice with y fixed is shown to the left, where $-1<y<1$.

(12) 4. Compute $\iint_D e^{-x^2-y^2}dA$ where D is the region in the plane which is inside the unit circle (the circle with center at $(0,0)$ and radius 1) and also inside the upper half plane (where $y\ge0$).

Answer A picture of the region is to the right. It is *friendly* to polar coordinates. The integral is $\int_0^\pi\int_0^1 e^{-r^2}r\,dr\,d\theta=\int_0^\pi -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=1}d\theta=\int_0^\pi\left(-\frac{1}{2}e^{-1}+\frac{1}{2}\right)d\theta=\frac{\pi}{2}\left(1-\frac{1}{e}\right)$.

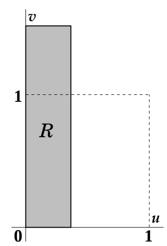


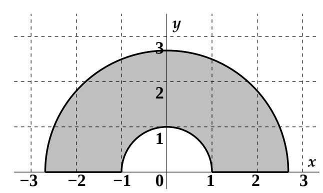
(12) 5. Express in cylindrical coordinates and evaluate: $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}} z\,dz\,dy\,dx$.
Answer $z\,dz\,dy\,dx$ becomes $zr\,dr\,d\theta\,dz$. The boundary $z = \sqrt{x^2+y^2}$ becomes $z = r$. The boundary $y = \sqrt{1-x^2}$ along with the knowledge that x goes from 0 to 1 describes the part of the unit disc in the first quadrant (similar to the setup of the previous problem) because $y = \sqrt{1-x^2}$ is part of $x^2+y^2 = 1$, and $\sqrt{\ }$ is always *non-negative* square root. Since the z boundary description involves r, I will change the order from $zr\,dr\,d\theta\,dz$ to $zr\,dz\,d\theta\,dr$. The triple integral becomes $\int_0^1 \int_0^{\pi/2} \int_0^r zr\,dz\,d\theta\,dr = \int_0^1 \int_0^{\pi/2} \frac{rz^2}{2}\Big|_{z=0}^{z=r} d\theta\,dr = \int_0^1 \int_0^{\pi/2} \frac{r^3}{2}\,d\theta\,dr = \int_0^1 \frac{r^3}{2}\theta\Big|_{\theta=0}^{\theta=\pi/2} dr = \int_0^1 \frac{\pi r^3}{4}\,dr = \frac{\pi r^4}{16}\Big|_{r=0}^{r=1} = \frac{\pi}{16}$

(12) 6. Use spherical coordinates to calculate the triple integral of $f(x,y,z) = x^2+y^2+z^2$ over the region $1 \le x^2+y^2+z^2 \le 4$.
Answer $\rho^2 = x^2+y^2+z^2$ so the region is just $1 \le \rho \le 2$ with all θ's ($0 \le \theta \le 2\pi$) and all ϕ's ($0 \le \phi \le \pi$. The integrand in spherical coordinates is ρ^2. So the desired triple integral is $\int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho^2\left(\rho^2 \sin\phi\right) d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \rho^4(\sin\phi)\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5}(\sin\phi)\Big|_{\rho=1}^{\rho=2} d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{31}{5}(\sin\phi)\,d\phi\,d\theta = \int_0^{2\pi} \frac{31}{5}(-\cos\phi)\Big|_{\phi=0}^{\phi=\pi} d\theta$ $= \int_0^{2\pi} \frac{31}{5}\big(-\cos\pi - (-\cos 0)\big)\Big|_{\phi=0}^{\phi=\pi} d\theta = \int_0^{2\pi} 2\left(\frac{31}{5}\right) d\theta = \frac{62}{5}\theta\Big|_{\theta=0}^{\theta=2\pi} = \frac{124\pi}{5}$, although $\frac{(2^5-1)4\pi}{5}$ is simpler.

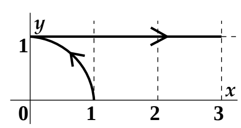
(12) 7. This problem is about the transformation $\begin{cases} x = e^{3u}\cos(2v) \\ y = e^{3u}\sin(2v) \end{cases}$.

a) Compute the Jacobian of this transformation. The result should be $6e^{6u}$ but you must show the details of the computation. **Answer** We need $\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \det\begin{pmatrix} 3e^{3u}\cos(2v) & -2e^{3u}\sin(2v) \\ 3e^{3u}\sin(2v) & 2e^{3u}\cos(2v) \end{pmatrix} =$
$\big(3e^{3u}\cos(2v)\big)\big(2e^{3u}\cos(2v)\big) - \big(-2e^{3u}\sin(2v)\big)\big(3e^{3u}\sin(2v)\big) =$
$6e^{6u}\big(\cos(2v)\big)^2 + 6e^{6u}\big(\sin(2v)\big)^2 = 6e^{6u}\Big(\big(\cos(2v)\big)^2 + \big(\sin(2v)\big)^2\Big)$
and this is $6e^{6u}$.
b) Suppose R is the region in the uv-plane determined by $u = 0$, $u = \frac{1}{3}$, $v = 0$, and $v = \frac{\pi}{2}$ as shown on the coordinate axes below and to the left. Sketch the image region using this transformation in the xy-plane below and to the right.





(16) 8. a) Compute $\int_C x\,dx + y^2\,dy$ if C is a quarter circle centered at $(0,0)$ from $(1,0)$ to $(0,1)$ followed by a line segment from $(0,1)$ to $(3,1)$. C is shown in a diagram to the right. You may need more than one integral!



Answer From $(0,0)$ to $(0,1)$ use $x = \cos t$ and $y = \sin t$ so $dx = -\sin t\,dt$, $dy = \cos t\,dt$, and $0 \le t \le \frac{\pi}{2}$. The integral over that portion of the curve is $\int_0^{\pi/2} -(\cos t)(\sin t) + (\sin t)^2(\cos t)\,dt = -\frac{(\sin t)^2}{2} + \frac{(\sin t)^3}{3}\Big|_0^{\pi/2} = -\frac{1}{2} + \frac{1}{3} = -\frac{1}{6}$. For the line segment, $x = t$ and $y = 1$ so $dx = dt$ and $dy = 0\,dt$, and $0 \le t \le 3$. So this integral is $\int_0^3 t\,dt = \frac{9}{2}$. The total integral is therefore $\frac{9}{2} - \frac{1}{6} = \frac{13}{3}$.
Another method $\varphi(x,y) = \frac{x^2}{2} + \frac{y^3}{3}$ is a potential for $x\mathbf{i} + y^2\mathbf{j}$ (verify this by checking φ using partial differentiation). Then the integral is $\varphi(\textbf{The end}) - \varphi(\textbf{The start}) = \varphi(3,1) - \varphi(1,0) = \left(\frac{3^2}{2} + \frac{1}{3}\right) - \frac{1}{2} = \frac{13}{3}$.
b) Suppose $\mathbf{F}$ is the vector field $(x+5y^2)\mathbf{i} + (Axy)\mathbf{j}$, where A is a constant. There is one value of A for which this vector field is a gradient vector field. Find that value of A. Then find all potentials of $\mathbf{F}$, using that value of A. **Answer** $\frac{\partial}{\partial y}$ of $x+5y^2$ is $10y$, and $\frac{\partial}{\partial x}$ of Axy is Ay, so the desired value of A is 10. Now $\int x+5y^2\,dx = \frac{x^2}{2} + 5xy^2 + C_1(y)$ and $\int 10xy\,dy = 5xy^2 + C_2(x)$ where $C_1(y)$ and $C_2(x)$ are unknown functions. But inspection of the two descriptions of the potential tells me that the most general potential of $\mathbf{F}$ is $\frac{x^2}{2} + 5xy^2 + C$ for any constant C.

Brief answers to version B

1. $x+5yz$ has answer $\pm\frac{13}{5}$. 2. $I = \int_0^2\int_{x^3}^9 xy\,dy\,dx = 65$. The first partial integration has answer $\frac{81x}{2} - \frac{x^7}{2}$. The graph is similar, and c)'s answer is $\int_8^9\int_0^2 xy\,dx\,dy + \int_0^8\int_0^{y^{1/3}} xy\,dx\,dy$. 3. The same. 4. $\int_\pi^{2\pi}\int_0^1 e^{-r^2} r\,dr\,d\theta$ with the same answer. 5. The same. 6. The answer is $\frac{(3^5-1)4\pi}{5} = \frac{968\pi}{5}$. 7. a) A similar computation gives the stated answer. The graph in b) is the same. 8. a) Much the same parameterizations can be used. The answer is $\frac{16}{2} - \frac{1}{6} = 8 - \frac{1}{6} = \frac{47}{6}$. In b), $A=6$ and the potential is $\frac{x^2}{2} + 3xy^2 + C$ for any constant C.