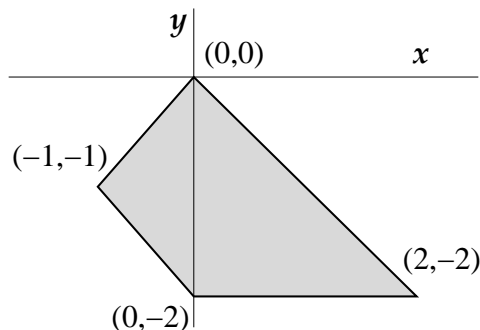(18)   1. Compute $\iint_R 2x + y \, dA$ if R is the region in $\mathbb{R}^2$ shown to the right. The boundary of R is four line segments meeting at $(0,0)$, $(-1,-1)$, $(0,-2)$, and $(2,-2)$.


(20)   2. Suppose P is the point $(2, 1, 6)$ in $\mathbb{R}^3$.
Suppose f is the function from $\mathbb{R}^3$ to $\mathbb{R}$ defined by $f(x, y, z) = (z - x^2)e^{3xy - z}$.
Suppose C is the curve in $\mathbb{R}^3$ defined by $C(t) = (3 - t^2, -\cos(\pi t), 3 + 3t)$.

a) Verify by computation that P is on the level surface $f(x, y, z) = 2$. Verify by computation that $C(1)$ is P.

b) Compute ∇f and evaluate it at P. Compute $C'(t)$ and evaluate it when $t = 1$.

c) Find an equation for the plane tangent to $f(x, y, z) = 2$ at the point P. Find parametric equations for the line tangent to the curve $C(t)$ when $t = 1$.

d) The plane and line found in c) intersect at P. Are they perpendicular? Give a reason for your answer.

(14)   3. Suppose that $G(u, v)$ is a differentiable function of two variables and that $g(x, y) = G\left(\frac{x}{y}, \frac{y}{x}\right)$. Show that $xg_x(x, y) + yg_y(x, y) = 0$.
Comment Your work must refer to a general function, G. Using one specific G or a few specific G's earns no credit.

(16)   4. Compute $\iint_Q \frac{1}{(x^2 + y^2 + 1)^3} \, dA$ where Q is the region in the first quadrant outside the unit circle.

(20)   5. A solid hemisphere (half of a solid sphere) of radius R has variable density δ proportional to the distance from the center of the hemisphere squared. Find its mass. (Be sure that you set up the problem correctly. Define clearly any constants you use, and describe clearly, using a picture if you like, where the hemisphere is located in your coordinate system.)

(20)   6. Suppose $\mathbf{F}(x, y, z) = (y^2 z) \mathbf{i} + (2xyz) \mathbf{j} + (xy^2 + 4z) \mathbf{k}$, a vector field defined and continuously differentiable throughout $\mathbb{R}^3$.

a) Determine whether there is a scalar function $P(x, y, z)$ defined everywhere in $\mathbb{R}^3$ such that $\nabla P = \mathbf{F}$. If there is such a P, find it; if there is not, explain why not.

b) Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{s}$, where C is the elliptical helix whose position vector is given by $\mathbf{R}(t) = (3 \cos t) \mathbf{i} + (2 \sin t) \mathbf{j} + 5t \mathbf{k}$ for $0 \le t \le 2\pi$. Use information gotten from your answer to a) to help if you wish. Some methods are easier than others.

(16)   7. Find all critical points of $f(x, y) = (x^2 + y^2) e^{x - y}$. Describe (as well as you can) the type of each critical point. Explain your conclusions.

(20)   8. Suppose $f(x, y, z) = e^{2z}$.

a) Compute $\int_0^1 \int_0^x \int_0^y e^{2z}\, dz\, dy\, dx$.

b) Write the integral in a) as a sum of one or more iterated integrals in $dx\, dy\, dz$ order. You are *not* asked to evaluate your answer, only to set it up.

(16)   9. The two-dimensional parametric curve C is defined by the equations $\begin{cases} x = \sin(t^2) \\ y = \cos(t^2) \end{cases}$.

a) Compute the curvature of C and please **simplify** your result. Use the formulas supplied if you wish.

b) Explain geometrically the answer to a) which should be quite simple.

(20)   10. Suppose Q is the oriented boundary of the parallelogram in $\mathbb{R}^3$ with corners at $A = (0, 0, 0)$, $B = (4, 1, 5)$, $C = (4, 0, 7)$, and $D = (0, -1, 2)$. The orientation of Q is from A to B to C to D and then back to A. A labeled `Maple` picture of Q is shown. Suppose $\mathbf{F}$ is the vector field defined by $\mathbf{F}(x, y, z) = \left(e^{(x^3)} + 7y\right)\mathbf{i} + \left(5x + \arctan(y^2)\right)\mathbf{j} + \left(z^{17} + 3y\right)\mathbf{k}$. Compute $\int_Q \mathbf{F} \cdot d\mathbf{s}$. Some methods are easier than others.
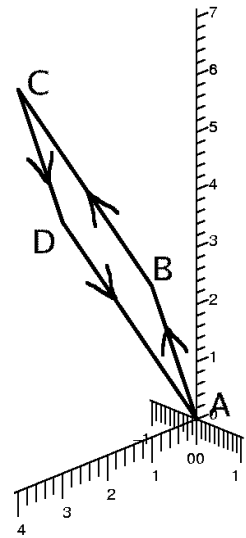
(12)   11. Suppose F is the vector field defined by $F(x, y, z) = \langle y^7 + x^2, \cos(x^3 + z^5) + y, e^{77x} + z\rangle$ and S is the boundary surface of the unit cube in $\mathbb{R}^3$, $0 \le x \le 1$ and $0 \le y \le 1$ and $0 \le z \le 1$, with *outward normal* orientation. Find the total flux of F through S. Some methods are easier than others.

(8)   12. Suppose $f(x, y) = xy^2$. On the axes supplied below [TO THE RIGHT], sketch the level curve which goes through $P = (1, 2)$. Label the curve with the appropriate function value. Also sketch on the same axes the gradient vector ∇f at P.
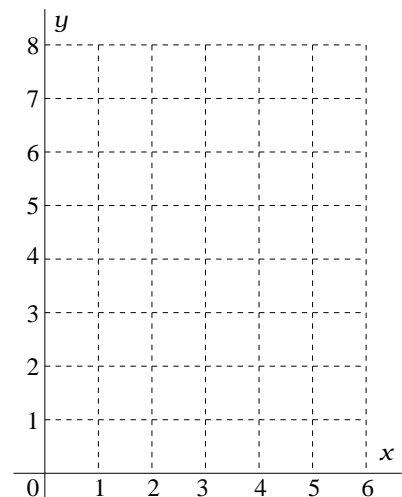
Just a few formulas for the final exam in Math 251, fall 2008

Curvature κ is all of the following:

$$\left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \overset{2 \text{ dim}}{=\!=} \frac{|y''(t)x'(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}} \overset{y=f(x)}{=\!=} \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

Second derivative test for differentiable functions in $\mathbb{R}^2$

Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let $H = H(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

    a) If $H > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

    b) If $H > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

    c) If $H < 0$, then $f(a, b)$ is not a local maximum or minimum (f has a saddle point).

If $H = 0$, no information.

Polar coordinates $dA = r \, dr \, d\theta$

Spherical coordinates $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Change of variables in 2 dimensions

$$\iint_{R_{xy}} f(x, y) \, dA = \iint_{\tilde{R}_{uv}} f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du \, dv \text{ where the Jacobian, } \frac{\partial(x, y)}{\partial(u, v)}$$

is $\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$.

Green's Theorem

$$\int_C P \, dx + Q \, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

If $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ and $\mathbf{F}$ is a vector field then $\begin{cases} \text{curl } F = \nabla \times \mathbf{F}, \text{ a vector field.} \\ \text{div } F = \nabla \cdot \mathbf{F}, \text{ a function.} \end{cases}$

Stokes' Theorem

S is a surface with boundary curve C. As you "walk" along C, S is to the left and $\mathbf{N}$, the surface normal, is up.

$$\left[\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{N} \, dS = \right] \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{s} \left[= \int_C P \, dx + Q \, dy + R \, dz\right]$$

Divergence Theorem

W is a region in $\mathbb{R}^3$ with boundary surface S. The boundary S is oriented so its normal vectors point *outward*.

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } F \, dV \left[= \iiint_E \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \, dV\right]$$

Final Exam for Math 251, sections 22–24

December 15, 2008

NAME ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

Do all problems, in any order.

Show your work. An answer alone may not receive full credit.

No notes and no calculators may be used on this exam.

A sheet with some formulas is the last page of the exam

"Simplification" of answers is not necessary <u>unless otherwise</u> <u>stated</u>, but standard values of traditional functions such as e^0 and $\sin\left(\frac{\pi}{2}\right)$ should be given.

Problem Number	Possible Points	Points Earned:
1	18	
2	20	
3	14	
4	16	
5	20	
6	20	
7	16	
8	20	
9	16	
10	20	
11	12	
12	8	
Total Points Earned:		