(20) 1. In this problem, suppose that $f(x,y,z)=\frac{x^3-2yz}{y^2+xz}$. Notice that $f(-1,1,2)=5$.*

a) Find $\nabla f(x,y,z)$. Compute $\nabla f(-1,1,2)$ which you may wish to simplify.

Answer $f_x=\frac{3x^2(y^2+xz)-z(x^3-2yz)}{(y^2+xz)^2}$, $f_y=\frac{-2z(y^2+xz)-2y(x^3-2yz)}{(y^2+xz)^2}$, $f_z=\frac{-2y(y^2+xz)-x(x^3-2yz)}{(y^2+xz)^2}$. $\nabla f(x,y,z)=\langle f_x,f_y,f_z\rangle$. At $(-1,1,2)$, the denominator of all the terms is $(-1)^2$, so things are not too horrible. ∇f is $\langle 7,14,-3\rangle$.

b) b) Write the equation of a plane tangent to $\frac{x^3-2yz}{y^2+xz}=5$ $f(x,y,z)=5$ at the point $(-1,1,2)$.

Answer $7(x-(-1))+14(y-1)-3(z-2)=0$.

c) Write parametric equations for a line normal to $f(x,y,z)=5$ at the point $(-1,1,2)$.

Answer $x=7t-1, y=14t+1, z=-3t+2$.

d) Find the directional derivative of f in the direction of the unit vector $\langle -\frac{1}{\sqrt6},\frac{2}{\sqrt6},\frac{1}{\sqrt6}\rangle$ at the point $(-1,1,2)$.

Answer $7\left(-\frac{1}{\sqrt6}\right)+14\left(\frac{2}{\sqrt6}\right)-3\left(\frac{1}{\sqrt6}\right)$

e) Find a unit vector in the direction of the largest directional derivative of f at the point $(-1,1,2)$.

Answer Since $7^2+14^2+(-3)^2=49+196+9=254$, the answer is $\langle\frac{7}{\sqrt{254}},\frac{14}{\sqrt{254},}\frac{-3}{\sqrt{254}}\rangle$.

f) What is the value of the largest directional derivative of f at the point $(-1,1,2)$?

Answer $\sqrt{254}$.

(14) 2. Suppose that x^2+px+q has roots r and s. $\boxed{x^2+x-6=(x-2)(x+3)}$

a) Write formulas for r and s as functions of p and q. (Nothing more is asked here: only "high school algebra".)

Answer The roots are $-p\pm\sqrt{p^2-4q}$, so $r=\frac{-p+\sqrt{p^2-4q}}{2}$ and $s=\frac{-p-\sqrt{p^2-4q}}{2}$.

b) Verify that the functions found in a) give 2 and -3 for r and s if $p=1$ and $q=-6$.

Answer $r=\frac{-1+\sqrt{1^2-4(-6)}}{2}=\frac{-1+\sqrt{25}}{2}=\frac{4}{2}=2$ and $s=\frac{-1-\sqrt{1^2-4(-6)}}{2}=\frac{-1-\sqrt{25}}{2}=\frac{-6}{2}=-3$.

c) Suppose p changes from 1 to 1.03 and q, from -6 to -6.04. Use linear approximation applied to the functions found in a) to find the approximate changes in the roots r and s.

Answer $\triangle r=r_p\triangle p+r_q\triangle q$. Here $r_p=\frac{-1+\frac12(p^2-4q)^{-1/2}2p}{2}$ and $r_q=\frac{\frac12(p^2-4q)^{-1/2}(-4)}{2}$. When $p=1$ and $q=-6$, $r_p=-\frac25$ and $r_q=-\frac15$ so $\triangle r=-\frac25(.03)-\frac15(-.04)=-.004$. `Maple` tells me that one root of the modified quadratic is ≈ 1.99602, so this $\triangle r$ looks good.

$\triangle s=s_p\triangle p+s_q\triangle q$. Here $s_p=\frac{-1-\frac12(p^2-4q)^{-1/2}2p}{2}$ and $s_q=-\frac{\frac12(p^2-4q)^{-1/2}(-4)}{2}$. When $p=1$ and $q=-6$, $r_p=-\frac35$ and $r_q=\frac15$ so $\triangle r=-\frac35(.03)+\frac15(-.04)=-.026$. `Maple` tells me that the other root is ≈ -3.02602, so $\triangle s$ is also good.

(16) 3. a) Find an equation of the plane through $(4,1,-2)$ which contains the line $\mathbf{r}(t)=\langle 4,1,6\rangle+t\langle 1,4,1\rangle$.

Answer The vector from $(4,1,-2)$ to $(4,1,6)$ is $\langle 0,0,8\rangle$ and $\langle 0,0,8\rangle\times\langle 1,4,1\rangle$ is $-32\mathbf{i}+8\mathbf{j}$, which is a vector perpendicular to the plane we want. So: $-32(x-4)+8(y-1)=0$.

b) The plane found in a) and the line $\mathbf{s}(t)=\langle -2,0,3\rangle+t\langle 3,1,1\rangle$ intersect. Find the point of intersection.

Answer For the new line, $x=-2+3t$, $y=0+1t$, and $z=3+t$. This is on the plane $-32(x-4)+8(y-1)=0$ when $-32\big((-2+3t)-4\big)+8(t-1)=0$ or $-88t+56=0$ so $t=\frac{7}{11}$. The point is $(-2+3\left(\frac{7}{11}\right),\frac{7}{11},3+\left(\frac{7}{11}\right))$.

(12) 4. If $x=s^2-t^2$, $y=2st$, and $z=f(x,y)$, show that $\left(\frac{\partial z}{\partial s}\right)^2+\left(\frac{\partial z}{\partial t}\right)^2=4\sqrt{x^2+y^2}\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)$.

Answer The Chain Rule says that $z_s=z_xx_s+z_yy_s=z_x2s+z_y2t$ and $z_t=z_xx_t+z_yy_t=z_x(-2t)+z_y2s$. Therefore $(z_s)^2+(z_t)^2=(z_x2s+z_y2t)^2+(z_x(-2t)+z_y2t)^2=(z_x)^2\,4s^2+8st+(z_y)^2\,4t^2\,(z_x)^2\,4t^2-8st+(z_y)^2\,4s^2=4(s^2+t^2)\left((z_x)^2+(z_y)^2\right)$, Since $x=s^2-t^2$ and $y=2st$, $x^2+y^2=s^4-2s^2t^2+t^4+4(st)^2=s^4+2s^2t^2+t^4=(s^2+t^2)^2$, and $\sqrt{x^2+y^2}=s^2+t^2$.

(8) 5. Find $\lim_{(x,y)\to(0,0)}\frac{xy\cos y}{3x^2+y^2}$ or show that the limit does not exist.

Answer When $x=0$, $\frac{xy\cos y}{3x^2+y^2}$ becomes 0. So if the limit exists, its value should be 0. When $y=0$, the quotient is also 0. But if $x=y=t$, then the quotient becomes $\frac{t^2\cos t}{4t^2}$ which is $\frac{\cos t}{4}$. As $t\to 0$, this $\to\frac14$. The limit does not exist.

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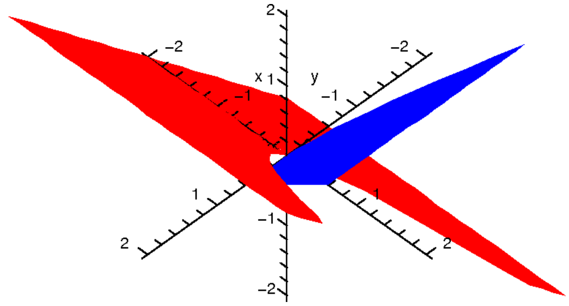
* Yes: $\frac{(-1)^3-2(1)(2)}{1^2+(-1)2}=\frac{-1-4}{1-2}=\frac{-5}{-1}=5$.

(12) 6. Suppose the function $f(x, y)$ with domain all of $\mathbb{R}^2$ is defined by $f(x,y) = \begin{cases} y & \text{if } y > x^2 \\ x & \text{if } y \le x^2 \end{cases}$.

a) Sketch a graph of $z = f(x, y)$. (You may wish to sketch two graphs and assert that your answer is a combination of these two!)

Answer Here, with some effort, is a `Maple` graph of this function. `Maple` does allow functions defined "piecewise" (try `help(piecewise)`!). A direct `plot3d` command of the `piecewise` function gives a rather poor result (try it!) because `Maple` does not handle "discontinuous" surfaces well. (It has more success in two dimensions, where the option `discont=true` can be used). What's shown is two different three-dimensional plots displayed together.



b) For which points (x, y) is $f(x, y)$ continuous? Consider all possible points in the domain, $\mathbb{R}^2$. Give some explanations for your answers. **Answer** Certainly *inside* each region with the $y = x^2$ *removed* the function is continuous. Thus, if (x, y) satisfies $y < x^2$ or, respectively, $y > x^2$, then "locally" (in a small disc around the point) $f(x, y)$ is x, respectively y. Polynomials are continuous everywhere. Where else can this function be continuous? Of course, the needed equation is $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$. This is true at two points on the curve $y = x^2$ because the values of y and x agree at these two points! Where are the equations $y = x$ and $y = x^2$ both true? The points are $(0, 0)$ and $(1, 1)$. Everywhere else on the parabola the values of y and x disagree, and the limit itself does not exists. So the function is *not* continuous at $y = x^2$ for $x \ne 0$ and $x \ne 1$, but it *is* continuous at $(0, 0)$ and $(1, 1)$.

(10) 7. A particle has position vector given by $\mathbf{R}(t) = \frac{1}{t}\mathbf{i} + t^2\mathbf{j} - 3t\mathbf{k}$.

a) What are the velocity and acceleration vectors of this particle when $t = 1$?

Answer $\mathbf{v}(t) = -\frac{1}{t^2}\mathbf{i} + 2t\mathbf{j} - 3\mathbf{k}$ so $\mathbf{v}(1) = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Also, $\mathbf{a}(t) = \frac{2}{t^3}\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$ so $\mathbf{a}(1) = 2\mathbf{i} + 2\mathbf{j}$.

b) Write the acceleration vector when $t = 1$ as a sum of two vectors, one parallel to the velocity vector when $t = 1$ and one perpendicular to the velocity vector when $t = 1$.

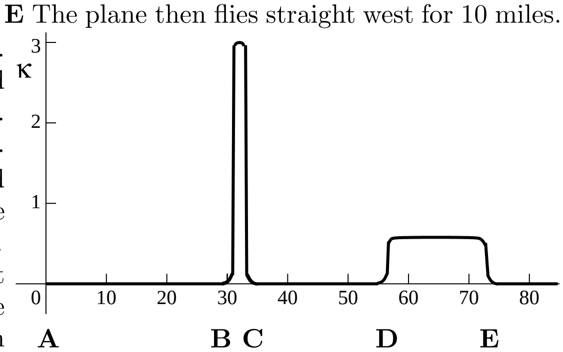
Answer $|\mathbf{v}(1)| = \sqrt{1 + 4 + 9} = \sqrt{14}$, and $\mathbf{a}(1) \cdot \mathbf{v}(1) = -2 + 4 = 2$ so that $\mathbf{a}_{\|} = \frac{\mathbf{a}(1)\cdot\mathbf{v}(1)}{|\mathbf{v}(1)|^2}\mathbf{v}(1) = \frac{2}{14}(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$. Normal component: $\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_T = 2\mathbf{i} + 2\mathbf{j} - \frac{2}{14}(-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$. A check: $\mathbf{a}_\perp \cdot \mathbf{v}(1) = \left(\frac{15}{7}\mathbf{i} + \frac{12}{7}\mathbf{j} + \frac{3}{7}\mathbf{k}\right) \cdot (-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = -\frac{15}{7} + \frac{24}{7} - \frac{9}{7} = 0$ so that the "normal" component is perpendicular to the velocity vector, as it's supposed to be.

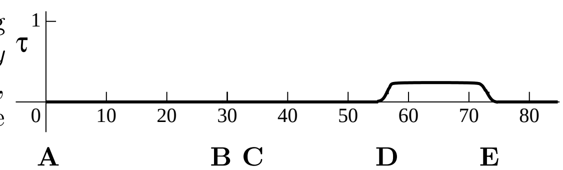
(8) 8. The flight of an airplane is described in this paragraph:

A The plane flies straight north for 30 miles. **B** The plane then makes a level quarter circular turn of radius $\frac{1}{3}$ mile. There is no change in altitude. **C** The plane then flies straight east for 20 miles. **D** The plane then gains altitude, flying on a right circular helical curve which has base radius 2 miles. The plane flies one and half loops of the helix, and has a 5 mile increase in altitude. **E** The plane then flies straight west for 10 miles.

Early thoughts The length of $\mathbf{B} \approx \frac{1}{4}\cdot 2\pi \cdot \frac{1}{3} \approx \frac{1}{2}$ and $\kappa = 3$. In **D** we have $x = 2\cos t$, $y = 2\sin t$, and $z = ?t$. "One and a half loops" means t goes from 0 to 3π. z goes from 0 to 5. When $t = 3\pi$, $z = 5$. Thus $?(3\pi) = 5$. Since $3\pi \approx 10$, $? \approx \frac{1}{2}$. For the helix, $\kappa = \frac{a}{a^2+b^2}$ and $\tau = \frac{b}{a^2+b^2}$. Since $a = 2$ and $b \approx .5$, $\kappa \approx .47$ and $\tau \approx .12$. Student graphs need *not* have such details! **D**'s length is a bit more than $1.5 \cdot 2\pi \cdot 2 \approx 20$.

a) Sketch a graph of the curvature, κ, of the plane flight as a function of the distance the plane has traveled. Write on the horizontal axis the letters **A**, **B**, **C**, **D**, and **E** when the plane is beginning the part of the flight corresponding to the description above. The graph should be *qualitatively* correct. Although exact numerical results are *not* needed, the vertical axis shown is probably sufficient to answer the question completely.





b) [SAME INSTRUCTIONS AS a) FOR torsion τ]