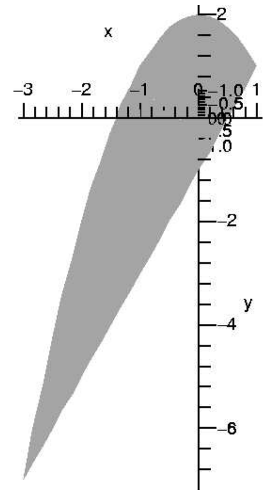
640:291:01 **Answers to the Second Exam** **11/26/2006**

(15) 1. This problem analyzes $\int_{-3}^{1}\int_{2x-1}^{2-x^2} x^2\,dy\,dx$.



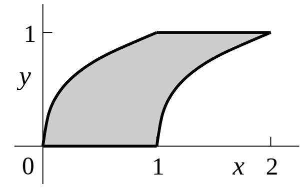
a) Compute $\int_{-3}^{1}\int_{2x-1}^{2-x^2} x^2\,dy\,dx$.

Answer $\int_{-3}^{1}\int_{2x-1}^{2-x^2} x^2\,dy\,dx = \int_{-3}^{1} x^2y\big]_{y=2x-1}^{y=2-x^2}\,dx = \int_{-3}^{1} x^2(2-x^2)-x^2(2x-1)\,dx = \int_{-3}^{1} 2x^2 - x^4 - 2x^3 + x^2\,dx = \int_{-3}^{1} 3x^2 - 2x^3 - x^4\,dx = x^3 - \frac{1}{2}x^4 - \frac{1}{5}x^5\big]_{x=-3}^{x=1} = \left(1-\frac{1}{2}-\frac{1}{5}\right) - \left((-3)^3 - \frac{1}{2}(-3)^4 - \frac{1}{5}(-3)^5\right)$. If you must "simplify" which, if not required, I would *discourage* in han d computation, this answer is also $\frac{96}{5}$.

b) Sketch the region in $\mathbb{R}^2$ over which the integral $\int_{-3}^{1}\int_{2x-1}^{2-x^2} x^2\,dy\,dx$ is computed on the axes provided. **Answer** A `Maple` graph is shown.

c) Write the integral $\int_{-3}^{1}\int_{2x-1}^{2-x^2} x^2\,dy\,dx$ as a sum of one or more integrals in $dx\,dy$ order. You are *not* asked to compute the result! **Answer** $2x-1 = 2-x^2$ when $x^2+2x-3=0$ so $(x+3)(x-1)=0$. When $x=-3$, $y=-7$, and when $x=1$, $y=1$. As for the boundary curves, if $y=2-x^2$ then $x=\pm\sqrt{2-y}$, and if $y=2x-1$, then $x=\frac{1}{2}(y+1)$. The answer: $\int_{y=-7}^{y=1}\int_{x=-\sqrt{2-y}}^{x=\frac{1}{2}(y+1)} x^2\,dx\,dx + \int_{y=1}^{y=2}\int_{x=-\sqrt{2-y}}^{x=\sqrt{2-y}} x^2\,dx\,dx$.

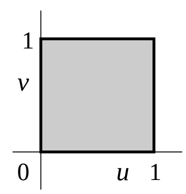
(15) 2. In this problem you will compute $\iint_R (x-y^2)^{100}\,dA$ where R is the region in $\mathbb{R}^2$ bounded by $y=0$, $y=1$, $y=\sqrt{x}$, and $y=\sqrt{x-1}$.



a) Sketch the region R on the axes provided.

b) Guess a transformation from (u,v) to (x,y) which will greatly simplify the integral.

Answer The "guess" of the *instructor* is $\begin{cases} u = x - y^2 \\ v = y \end{cases}$. That's because of the integrand, which involves a big power of $x-y^2$, and the boundary curve $y=\sqrt{x}$ which is $y^2=x$ or $x-y^2=0$, and the boundary curve $y=\sqrt{x-1}$ which is $x-y^2=1$.

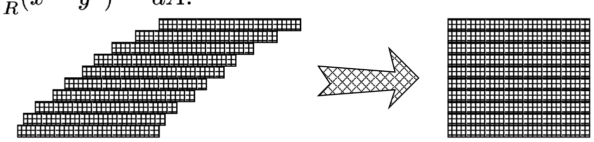


c) Sketch the region in (u,v) space on the axes provided which corresponds to the region R in (x,y) space.

d) Compute the Jacobian of the transformation from (u,v) to (x,y).

Answer Since $u=x-y^2$, we get $u=x-v^2$ so $x=u-v^2$ in addition to $y=v$. Now we compute the Jacobian. The matrix $\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ is $\begin{pmatrix} 1 & -2v \\ 0 & 1 \end{pmatrix}$ which has determinant $=1$.

e) Change variables from (x,y) to (u,v) and compute $\iint_R (x-y^2)^{100}\,dA$.



Answer $\int_0^1\int_0^1 u^{100}du\,dv = \frac{1}{101}$. **Comments** This problem had many correct answers, mostly like what's above. The change of variables is called a *non-linear shear*. The picture to the right may help you believe that the Jacobian, the area distortion factor, is 1. A similar result in three dimensions is called *Cavalieri's Principle* (the `Wikipedia` entry on Cavalieri's Principle has a wonderful picture of coins to justify the idea). I had hoped that most students would jump at this change of variables, and that the problem would be straightforward. By the way, I was surprised that direct `Maple` computation of the original iterated integral wasn't the "answer" – that is, after some effort the simple answer given above was gotten, but what was given originally was quite a bit more complicated, involving values of the Γ function and lots of other numbers.

Student suggestion # 1 (following Mr. Fitzgerald) Take $x=u^2$ and $y=v$. The (u,v) region then turns out to be bounded by $v=0$ and $v=1$, and also $u=v$ and $v=\sqrt{u^2-1}$. The Jacobian is $2u$, but the whole integral in u and v can be computed.

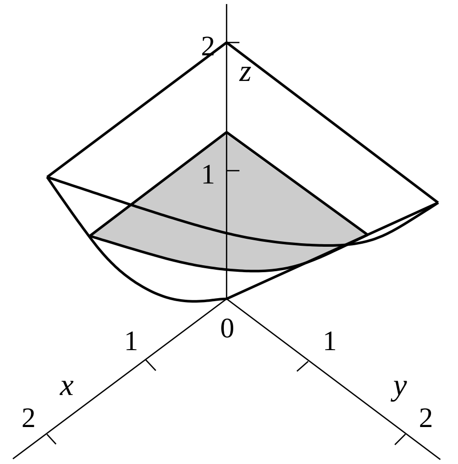
Student suggestion # 2 (Mr. Harvu and Mr. Mendat) Take $x=u$ and $y=v^2$. This does work out.

Student suggestion # 3 (Mr. Rind) Take $u=\frac{y^2}{x}$ and $v=y$. The region in (u,v) space isn't so nice.

Student suggestion # 4 (Mr. Kim and Mr. Stern) Take $u=\sqrt{x}+y$ and $v=\sqrt{x}-y$. I think this could work, but the region in (u,v) space seems quite complicated.

`Maple` **computation** A direct computation of the original integral (split into two iterated integrals) gives a result in terms of the Γ function, which can then be "simplified" into the answer gotten above.

(12) 3. Suppose R is the region in the first octant ($x \geq 0$, $y \geq 0$, and $z \geq 0$) bounded below by the surface $z = x^2 + y$ and is bounded above by the plane $z = 2$.



a) Sketch R on the axes provided. **Answer** There's a sketch to the right, also indicating a "typical" intermediate z slice.

b) Write the triple integral of xy over R as an iterated integral in $dx\,dy\,dz$ order. **Answer** Certainly z goes from 0 to 2. An intermediate z slice sits in the first quadrant of the xy plane. The curved boundary is $z = x^2 + y$. y's boundaries are 0 and z. Then x starts at 0 and goes "out" to $\sqrt{z-y}$. The iterated integral is $\int_{z=0}^{z=2}\int_{y=0}^{y=z}\int_{x=0}^{x=\sqrt{z-y}} xy\,dx\,dy\,dz$.

c) Compute $\iiint_R xy\,dx\,dy\,dz$. **Answer** We use Fubini's Theorem and compute the iterated integral. We begin: $\int_{x=0}^{x=\sqrt{z-y}} xy\,dx = \left.\frac{x^2y}{2}\right]_{x=0}^{x=\sqrt{z-y}} = \frac{(z-y)y}{2} = \frac{1}{2}zy - \frac{1}{2}y^2$. Then $\int_{y=0}^{y=z}\frac{1}{2}zy - \frac{1}{2}y^2\,dy = \left.\frac{zy^2}{4} - \frac{y^3}{6}\right]_{y=0}^{y=z} = \frac{1}{12}z^3$. Finally, $\int_{z=0}^{z=2}\frac{1}{12}z^3\,dz = \left.\frac{1}{48}z^4\right]_{z=0}^{z=2} = \frac{1}{3}$.

(14) 4. a) Suppose D is the unit ball in $\mathbb{R}^3$: those points in $\mathbb{R}^3$ whose distance to the origin is less than or equal to 1. If A is a non-negative real number ($A \geq 0$) compute the triple integral of $(x^2+y^2+z^2)^A$ over D. Your answer should depend on A. **Hint** The answer for $A = 0$ is well-known!

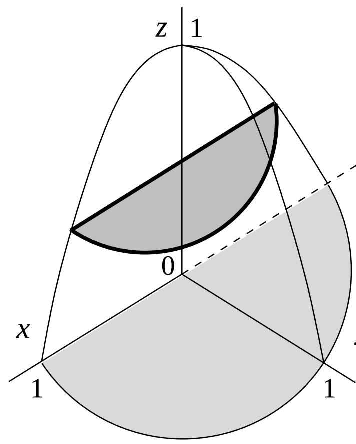
Answer In spherical coordinates, $(x^2+y^2+z^2)^A = (\rho^2)^A = \rho^{2A}$. The unit ball in spherical coordinates is easy but don't forget the Jacobian. So: $\int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^{2A}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^{2A+2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{4\pi}{2A+3}$. If $A = 0$ this is $\frac{4}{3}\pi$, the volume of the unit ball.

b) If $A < 0$ the integral of $(x^2+y^2+z^2)^A$ over D is officially an improper integral. Suppose $0 < s < 1$ and D_s is those points in $\mathbb{R}^3$ whose distance to the origin is between s and 1. Compute the triple integral of $(x^2+y^2+z^2)^A$ over D_s. Your answer should depend on both s and A. For which A's does the result approach a finite limit as $s \to 0^+$, and what is the limit?

Answer The limits for D_s in spherical coordinates differ from those for D only in the ρ part. The D_s limits in ρ are from s to 1. Therefore we have:
$\int_0^{2\pi}\int_0^{\pi}\int_s^1 \rho^{2A}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{4\pi}{2A+3}\left(1 - s^{2A+3}\right)$. If $2A+3 > 0$, then $s^{2A+3} \to 0$ as $s \to 0^+$. So if $A > -\frac{3}{2}$, the limit is finite, and its value is $\frac{4\pi}{2A+3}$. When $s < -\frac{3}{2}$, the term s^{2A+3} has a *negative* exponent, and it approaches $+\infty$ as $s \to 0^+$. To be complete, we should note that when $2A+3 = 0$, the integral will instead have a log term, and the log term $\to -\infty$ as $s \to 0^+$. There is convergence exactly when $s > -\frac{3}{2}$.

(12) 5. Suppose $f(x,y,z) = Ay + Bz$, and suppose R is the region in $\mathbb{R}^3$ which is contained in the half-space $y \geq 0$ and is bounded above by $z = 1 - x^2 - y^2$ and below by $z = 0$. Find rational numbers A and B so that $\iiint_R f(x,y,z)\,dV = 1 + \pi$.

Comment Rational numbers are quotients of integers. π is not a rational number. Be careful of the r's!



Answer The problem is perhaps best done in cylindrical coordinates. The integrand: $Ay + Bz$ becomes $Ar\sin\theta + Bz$. $y \geq 0$ translates to $0 \leq \theta \leq \pi$. And of course $z = 1 - x^2 - y^2$ is $z = 1 - r^2$, which is the same as $r = \sqrt{1-z}$. Slices of the solid perpendicular to the z-axis are semicircles. See the sketch to the right. The integral $\iiint_R f(x,y,z)\,dV$ is $\int_0^1\int_0^{\pi}\int_0^{\sqrt{1-z}}(Ar\sin\theta + Bz)r\,dr\,d\theta\,dz = \int_0^1\int_0^{\pi}\int_0^{\sqrt{1-z}}(Ar^2\sin\theta + Brz)\,dr\,d\theta\,dz$. The innermost integral is $\left.\frac{A}{3}r^3\sin\theta + \frac{B}{2}r^2z\right]_{r=0}^{r=\sqrt{1-z}} = \frac{A}{3}(1-z)^{3/2}\sin\theta + \frac{B}{2}(1-z)z$. Integrate this $d\theta$ and get $\frac{2A}{3}(1-z)^{3/2} + \frac{B\pi}{2}\left(z - z^2\right)$. Antidifferentiation with respect to z gives $\left.-\frac{4A}{15}(1-z)^{5/2} + \frac{B\pi}{2}\left(\frac{1}{2}z - \frac{1}{3}z^2\right)\right]_{z=0}^{z=1}$. This is all a bit tricky because one term of the antiderivative vanishes at one limit and the other term vanishes at the other limit. The answer is $\frac{4A}{15} + \frac{B\pi}{12}$. This will become $1 + \pi$ if $A = \frac{15}{4}$ and $B = 12$.

(16) 6. a) Find and classify (local max, local min, or neither) all critical points of $f(x,y) = (x+y)e^{2y-x^2}$.

Answer This is somewhat tedious. So: $f_x = e^{2y-x^2} + (x+y)e^{2y-x^2}(-2x) = (1-2x(x+y))e^{2y-x^2}$ and $f_y = e^{2y-x^2} + (x+y)e^{2y-x^2}2 = (1+2(x+y))e^{2y-x^2}$. The exponential function is never 0, so that critical points all occur where $\begin{cases} 1-2x(x+y) = 0 \\ 1+2(x+y) = 0 \end{cases}$. The second equation asserts that $x+y = -\frac{1}{2}$ and the first equation then becomes $1 - 2x\left(-\frac{1}{2}\right) = 0$ so that $x = -1$. Then $x+y = -\frac{1}{2}$ gives $y = \frac{1}{2}$. The only critical point is $\left(-1, \frac{1}{2}\right)$.

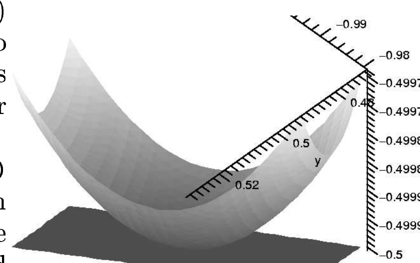
Now for the second derivative test. Since $f_x = (1-2x^2-2xy)e^{2y-x^2}$, $f_{xx} = (-4x-2y)e^{2y-x^2} + (1-2x^2-2xy)e^{2y-x^2}(-2x)$ and $f_{xy} = -2xe^{2y-x^2} + (1-2x(x+y))e^{2y-x^2}2$. At the critical point, $e^{2y-x^2} = e^{1-1} = e^0 = 1$, making things a bit easier. At $\left(-1, \frac{1}{2}\right)$, $f_{xx} = (4-1) + (1-2+1)(-2) = 3$ and $f_{xy} = 2 + (1+2(-\frac{1}{2}))2 = 2$ And since $f_y = (1+2(x+y))e^{2y-x^2}$, $f_{yx} = 2e^{2y-x^2} + (1+2(x+y))e^{2y-x^2}(-2x)$ and $f_{yy} = 2e^{2y-x^2} + (1+2(x+y))e^{2y-x^2}2$. At $\left(-1, \frac{1}{2}\right)$, $f_{yx} = 2$ and $f_{yy} = 2$.

By the way, I did do all of these calculations by hand but then I checked the results with `Maple`. And, by the way, when I do the calculations, I usually independently compute f_{yx} and f_{xy}. If the results are different, then I worry. Otherwise it is a (fairly) cheap way to check. The Hessian is $\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$ and the second derivative test asserts that the critical point is a local minimum.

Comment To the right is a `Maple` picture of the graph $z = f(x,y)$ near the critical point together with the (horizontal) tangent plane to the surface at that point. I needed to work diligently to get what's shown, since the function seems not to be very rapidly varying near $\left(-1, \frac{1}{2}\right)$. What's shown is the result of the command

```
plot3d({f,1/2},y=.5*(x^2-.02)...5*(x^2+.02),x=-.98..-1.02)
```

and the strange options in `plot3d` operate much like the limits on an iterated integral and identify a region in $\mathbb{R}^2$ over which to draw the graph, a region which is around $\left(-1, \frac{1}{2}\right)$ and near $y = x^2$, a twisted rectangle. A more conventional rectangle doesn't show the graph well.



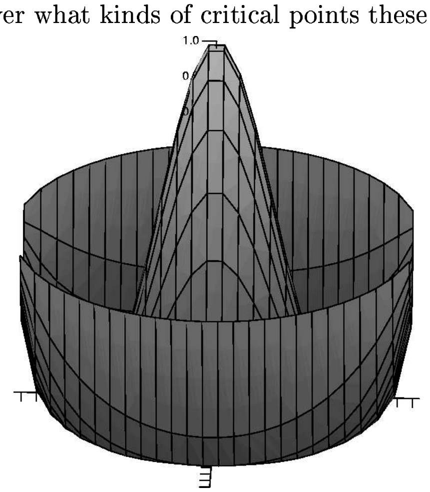
b) Find and classify (local max, local min, or neither) all critical points of $g(x,y) = (x^2+y^2-1)^{456}$.

Hint Compute a little bit and then *think*!

Answer $g_x = 456(x^2+y^2-1)^{455}2x$ and $g_y = 456(x^2+y^2-1)^{455}2y$. For which (x,y)'s are both $g_x = 0$ and $g_y = 0$? Certainly if $x^2+y^2-1 = 0$ (the unit circle) this happens. But also if the other factors (x and y) are both 0. So the origin, $(0,0)$, is also a critical point. Let's try to discover what kinds of critical points these are. Certainly if $x^2+y^2 = 1$, then $g(x,y) = (1-1)^{456} = 0$. But 456 is even and therefore $g(x,y) \geq 0$ for *all* (x,y) in $\mathbb{R}^2$. Every point on the unit circle is a <u>minimum</u> of g! What about $(0,0)$? Since $g(0,0) = 1^{456} = 1$ and nearby (x,y)'s all have $|x^2+y^2-1| < 1$, for all (x,y) close to $(0,0)$, $g(x,y) < g(0,0)$. Therefore $(0,0)$ is a <u>local maximum</u>.

Comment Using the Second Derivative Test on $g(x,y)$ gives *no information*: all four parts of the Hessian matrix are 0.

Here's a graph of something like $g(x,y)$. `Maple` essentially *refused* to graph $(x^2+y^2-1)^{456}$ at almost any point *outside* the unit circle (`Range too large`). The 456th power is too darn large. Things get exaggerated and what `Maple` displays is almost silly. Here is part of the graph of $(x^2+y^2-1)^4$ (only the fourth power!). You can see the qualitative aspects of what is reported above. I needed to experiment quite a bit to produce this graph. (A "hat" with a peak and a brim?)



(16) 7. a) Abstract mathematical theory declares that the function $f(x,y) = x(y+1)$ must attain a maximum value and a minimum value on the set of points in $\mathbb{R}^2$ satisfying $g(x,y) = x^2+y^2 = 1$. Find the maximum and minimum values using Lagrange multipliers.

$$\begin{cases} y+1=\lambda(2x) \\ x=\lambda(2y) \\ x^2+y^2=1 \end{cases}$$

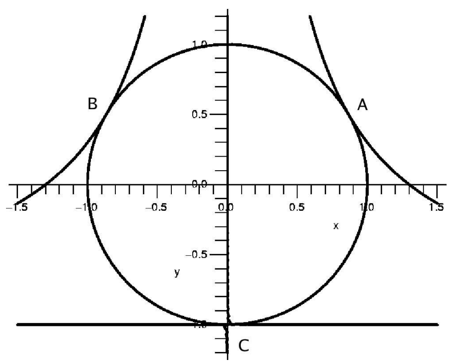
Answer The Lagrange multiplier equations (from the vector equation $\nabla f=\lambda\nabla g$ and the constraint equation) are shown to the right. Suppose we don't need to worry about division by 0 (!). The first two equations tell us $\lambda=\frac{y+1}{2x}=\frac{x}{2y}$ so $2y^2+2y=2x^2$ or $y^2+y=x^2$. Combine this with the constraint equation. We get $y^2+y=1-y^2$ so that $2y^2+y-1=0$. Then $y=\frac{-1\pm\sqrt{1-4(2)(-1)}}{2\cdot 2}=-\frac{1}{4}\pm\frac{3}{4}$. So y must be -1 or $\frac{1}{2}$, and then x is 0 or $\pm\frac{\sqrt{3}}{2}$, respectively. (Actually, I thought we had *excluded* the possibility of anything being 0!)

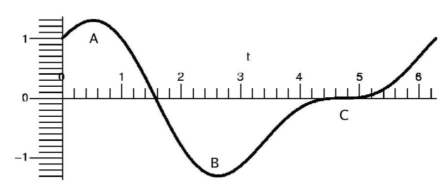
If we allow $x=0$, the first equation tells us that $y=-1$. If $y=0$, the second equation gives $x=0$, but $(0,0)$ is not on the constraint curve.

So the candidates for where max/min's occur are $(0,-1)$ and $\left(\pm\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. The values of the objective function, $x(y+1)$, at these points are 0 and $\pm\frac{3\sqrt{3}}{4}$. The last numbers are the maximum and minimum values predicted by theory for f.

Comments Here are some interesting pictures. The first picture, below to the left, shows the constraint and the level curves associated with the values 0 (at C) and $\pm\frac{3\sqrt{3}}{4}$ (at A and B). The level "curve" for 0 is actually not a very nice curve: it is two straight lines meeting perpendicularly, and the meeting point is on the constraint. Part of this level "curve" is inside the constraint and part of it is outside. I *bet* that this value of the objective function corresponds to an inflection point.

We can "unroll" the constraint in this problem. I mean we can parameterize the constraint curve with $x=\cos\theta$ and $y=\sin\theta$. Then the circle can be represented by the interval $[0,2\pi]$ (with the endpoints "identified"). The objective function, $x(y+1)$, becomes $\cos\theta(\sin\theta+1)$. The graph below on the right shows the objective function on the interval of interest. I hope you can see the maximum value of $\frac{3\sqrt{3}}{4}$ at A, the minimum value of $-\frac{3\sqrt{3}}{4}$ at B, and the inflection point associated with the value 0 at C.





b) Abstract mathematical theory declares that the function $f(x,y,s,t)=xy^2s^3t^4$ must attain a maximum value and a minimum value on the set of points in $\mathbb{R}^4$ satisfying $g(x,y,s,t)=x^2+y^2+s^2+t^2=1$. Find the maximum and minimum values using Lagrange multipliers.

$$\begin{cases} \frac{\partial}{\partial x}: y^2s^3t^4=\lambda 2x \\ \frac{\partial}{\partial y}: 2xys^3t^4=\lambda 2y \\ \frac{\partial}{\partial s}: 3xy^2s^2t^4=\lambda 2s \\ \frac{\partial}{\partial t}: 4xy^2s^3t^3=\lambda 2t \\ \text{Constraint}: x^2+y^2+s^2+t^2=1 \end{cases}$$

Answer I bet that the max and min are not attained where any of the variables are equal to 0, because then f's value will be 0. There are points satisfying the constraint equation with all coordinates not 0, so f is not always equal to 0.

The Lagrange multiplier equations are again shown to the right.

The first and second equations give: $\lambda=\frac{y^2s^3t^4}{2x}=\frac{2xys^3t^4}{2y}$, so $2y^3s^3t^4=4x^2ys^3t^4$ and $y^2=2x^2$.

The first and third equations give: $\lambda=\frac{y^2s^3t^4}{2x}=\frac{3xy^2s^2t^4}{2s}$, so $2y^2s^4t^4=6x^2y^2s^2t^4$ and $s^2=3x^2$.

The first and fourth equations give: $\lambda=\frac{y^2s^3t^4}{2x}=\frac{4xy^2s^3t^3}{2t}$, so $2y^2s^3t^5=8x^2y^2s^3t^3$ and $t^2=4x^2$.

The constraint equation $x^2+y^2+s^2+t^2=1$ becomes $10x^2=1$ at any critical point, and $x=\pm\frac{1}{\sqrt{10}}$. This allows us to get the other variables, and please note that the signs are unlinked, since the "communication" is through equations like $s^2=3x^2$.

Then $f(\text{a critical point})=\pm\frac{1}{\sqrt{10}}\sqrt{2}^2\left(\frac{1}{\sqrt{10}}\right)^2\sqrt{3}^3\left(\frac{1}{\sqrt{10}}\right)^3\sqrt{4}^4\left(\frac{1}{\sqrt{10}}\right)^4=\pm\frac{96\sqrt{3}}{100{,}000}$ and these numbers are the maximum and minimum values. There are 16 critical points, but a list of critical points is not requested.

Comment No pictures are shown. Sorry.