Math 291:01 **About the final exam** December 15, 2006

The final exam is on Friday, December 22, 2006, from 8 AM to 11 AM in SEC 205. I'd like to be *done* with the exam at 11 AM (!). Ten problems follow. The exam itself will consist of ten problems. Seven or eight of the exam problems will be selected from these problems. A formula sheet (separately linked on the web) will also be supplied. You may *not* bring any notes or calculators to use on the exam.

1. Find all critical points of each function. Describe (as well as you can) the type of each critical point. Explain your conclusions.

a) $f(x,y) = \left(x^2 + y^2\right) e^{x-y}$

b) $g(x,y) = \left(y - x^2\right)^{600}$

2. If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ and show that $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

3. The point $p = (-2,1,1)$ satisfies the equation $z^3 + xy^2z + 1 = 0$. Suppose near the point p that z is defined implicitly by the equation as a differentiable function of x and y.

a) If x is changed from -2 to -2.03 and y is changed from 1 to 1.04, use linear approximation to describe the approximate change in z.

b) What is the value of $\frac{\partial^2 z}{\partial x^2}$ at p?

4. Prove Green's Theorem for the region in the plane bounded by the x-axis and the curve $y = 1 - x^2$ by explicitly computing both sides of the equality for a "general" $P(x,y)\,dx + Q(x,y)\,dy$ (be sure to state what conditions on P and Q are needed) and checking that the two sides are indeed the same.

5. Suppose a vector field is defined by $\mathbf{F} = \left(y^2z\right)\mathbf{i} + (2xyz)\,\mathbf{j} + \left(xy^2 + 4z\right)\mathbf{k}$.

a) Determine whether there is a scalar function $P(x,y,z)$ defined everywhere in space such that $\nabla P = \mathbf{F}$. If there is such a P, find it; if there is not, explain why not.

b) Compute the integral $\int_W \mathbf{F}\cdot\mathbf{T}\,ds$, where W is the circular helix whose position vector is given by $\mathbf{R}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 2\pi$. Use information gotten from your answer to a) to help if you wish.

6. The average value of a function f defined in a region R of $\mathbb{R}^3$ is $\dfrac{\iiint_R f\,dV}{\iiint_R 1\,dV}$. Compute the average distance to the center of a sphere of radius a.

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7. Suppose $f(x,y,z) = xy^2z^3$.

a) Compute $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$.

b) Write the integral in a) as a sum of one or more iterated integrals in $dx\,dy\,dz$ order. You are *not* asked to integrate your answer, only to set it up.

8. Find the total flux upward through the upper hemisphere ($z \geq 0$) of the sphere $x^2 + y^2 + z^2 = a^2$ of the vector field $\mathbf{T} = \left(\frac{x^3}{3}\right)\mathbf{i} + \left(yz^2 + e^{\sqrt{z}}\right)\mathbf{j} + \left(zy^2 + y + 2 + \sin(x^3)\right)\mathbf{k}$.

Note Please *don't* compute this directly! Use the Divergence Theorem on some "simple" solid to change the desired computation to the computation of a triple integral and a much simpler flux integral. Evaluate those integrals, taking advantage of symmetry as possible.

9. Suppose $\mathbf{F} = -2xz\mathbf{i} + y^2\mathbf{k}$. **Note** There is *no* $\mathbf{j}$ component in $\mathbf{F}$.

a) Compute curl $\mathbf{F}$.

b) Compute the outward unit normal $\mathbf{n}$ for the sphere $x^2 + y^2 + z^2 = a^2$.

c) If R is any region on the sphere $x^2 + y^2 + z^2 = a^2$, verify that $\iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS = 0$.

d) Suppose C is a simple closed curve on the sphere $x^2 + y^2 + z^2 = a^2$. Show that the value of the line integral $\int_C -2xz\,dx + y^2\,dz$ is 0.

Comment Please *don't* attempt a direct computation! Use c) and one of the big theorems.

10. a) Verify that the improper integral $\int_0^1 x^{-3/2}\,dx$ does *not* converge.

b) Suppose R is the (roughly) triangular-shaped region in $\mathbb{R}^2$ defined by $y = x^2$, $y = 0$, and $x = 1$. For which values of a and b does the integral $\int\int_R x^a y^b\,dA$ converge?