Formula sheet for the first exam in Math 291, fall 2006

FIRST VERSION 3/2/2003: CORRECTED FROM FALL 2003; AMENDED 3/13/2003; A MORE CHANGE IN FALL 2006.

Cauchy-Schwarz: $|\mathbf{v}\cdot\mathbf{w}| \le \|\mathbf{v}\|\,\|\mathbf{w}\|$. Triangle inequality: $\|\mathbf{v}+\mathbf{w}\| \le \|\mathbf{v}\| + \|\mathbf{w}\|$.

Distance from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$ is $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$.

Distance from $P_1(x_1,y_1,z_1)$ to the plane $ax+by+cz=d$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\mathbf{n}=\langle a,b,c\rangle$

Line: $\{x=x_0+at, y=y_0+bt, z=z_0+ct\}$ through (x_0,y_0,z_0) in direction $\langle a,b,c\rangle$

$\|\mathbf{a}\| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

$|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$ (If $=0$, then $\mathbf{a}\perp\mathbf{b}$.) $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta$ (If $\mathbf{a}\|\mathbf{b}$, this $=0$.)

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ $\quad \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ $\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}$ $\quad \mathrm{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a}$

Volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $\|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\|$

Arc length: $\int_a^b \|\mathbf{r}'(t)\|\,dt$ $\quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ $\quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ $\quad \mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \overset{2\text{ dim}}{=} \frac{|y''(t)x'(t) - x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^{3/2}} \overset{y=f(x)}{=} \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

$\tau = \dfrac{(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2}$. $\quad$ Frenet-Serret: $\dfrac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\dfrac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}+\tau\mathbf{B}$, $\dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

Tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$: $z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

Linear approximation to $f(x,y)$ at (a,b): $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent plane to $F(x,y,z)=0$:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

If y implicitly defined by $y=f(x)$ in $F(x,y)=0$ then $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$.

If z implicitly defined by $z=f(x,y)$ in $F(x,y,z)=0$ then $z_x = -\dfrac{F_x}{F_z}$ and $z_y = -\dfrac{F_y}{F_z}$.

$\nabla f = \langle f_x, f_y, f_z\rangle = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$ $\quad D_{\mathbf{u}}f(x,y,z) = \nabla f(x,y,z)\cdot\mathbf{u}$

Some chain rules:

If $z=f(x,y)$ and $x=x(t)$ and $y=y(t)$, then $\dfrac{dz}{dt} = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}$.

If $z=f(x,y)$ and $x=g(s,t)$ and $y=h(s,t)$, then $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$.

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$. Let $H=H(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

a) If $H>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $H>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $H<0$, then $f(a,b)$ is not a local maximum or minimum (f has a saddle point).

A real-valued function $F(\mathbf{x})$ is continuous at $\mathbf{x_0}$ if, given any $\varepsilon>0$, there is a $\delta>0$ so that whenever $\|\mathbf{x}-\mathbf{x_0}\|<\delta$, then $|F(\mathbf{x})-F(\mathbf{x_0})|<\varepsilon$.