A triple integral

computed by iterated integrals two ways

The integrand ("mass density") is $5x+7y+11z$, and the region in $\mathbb{R}^3$ is the tetrahedron with vertices (corners) $(1,0,0)$, $(0,2,0)$, $(0,0,3)$, and $(0,0,0)$. I guessed at the equation for the tilted face of the tetrahedron: $x+\frac{1}{2}y+\frac{1}{3}z=1$.

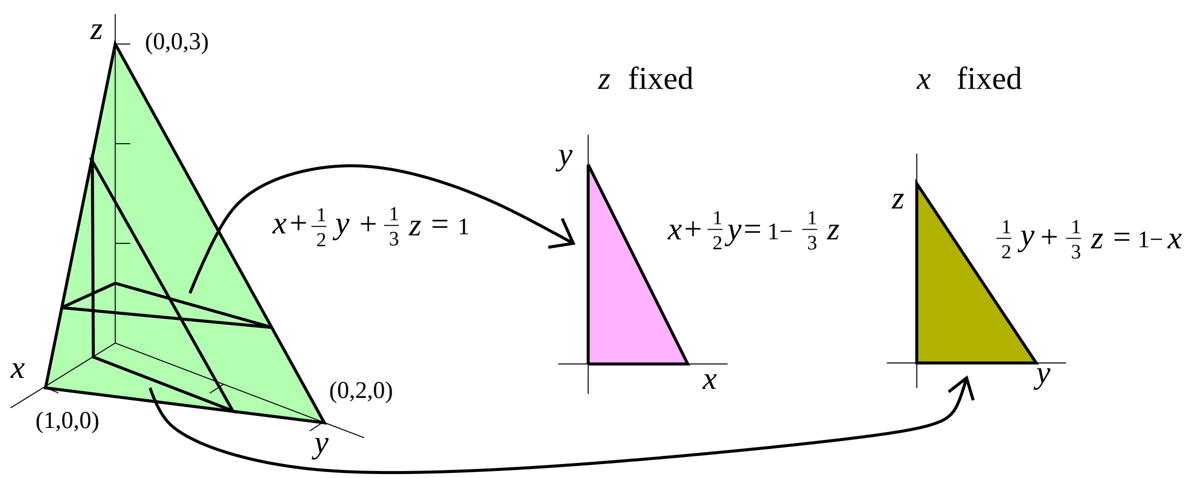
The mass is the triple integral over the region. So the mass is $\iiint_{\text{tetrahedron}} \text{density}\, dV$. We converted this into two different triply iterated integrals.

We first did the order $\underline{dx\,dy\,dz}$. Here I went from "outside in", and the limits on z went from 0 to 3. An intermediate z slice is shown in the middle picture. Then, going inside once more, the y limits are 0 and $2(1-x)$ (the intersection of the line in that picture with the y-axis). Finally, the x limits go from 0 to the line, so $x=1-\frac{1}{2}y-\frac{1}{3}z$.

The result Mass $=\int_{z=0}^{z=3}\int_{y=0}^{y=2(1-x)}\int_{x=0}^{x=1-\frac{1}{2}y-\frac{1}{3}z}(5x+7y+11z)\,dx\,dy\,dz$. A `Maple` computation of this is shown below. Look at `A`, `B`, and `C`.

Then we tried the order $\underline{dz\,dy\,dx}$. Here I went from "outside in", and the limits on x went from 0 to 1. An intermediate x slice is shown in the rightmost picture. Then, going inside once more, the y limits are 0 and $2(1-\frac{1}{3}z)$ (the intersection of the line in that picture with the y-axis). Finally, the z limits go from 0 to the line, so $z=3(1-\frac{1}{2}y-x)$.

The result Mass $=\int_{x=0}^{x=1}\int_{y=0}^{y=2(1-(1/3)z))}\int_{z=0}^{z=3(1-\frac{1}{2}y-x)}(5x+7y+11z)\,dz\,dy\,dx$. A `Maple` computation of this is shown below. Look at `A1`, `B1`, and `C1`.



These iterated integrals compute the mass in `Maple`.

```
> A:=int(5*x+7*y+11*z,z=0..3*(1-x-(1/2)*y));
A := 5*x*(3-3*x-3/2*y)+7*y*(3-3*x-3/2*y)+11/2*(3-3*x-3/2*y)^2
> B:=int(A,y=0..2*(1-x));
B := 5/8*(2-2*x)^3+1/2*(21*x-57/2)*(2-2*x)^2+5*x*(3-3*x)*(2-2*x)+11/2*(3-3*x)^2*(2-2*x)
> C:=int(B,x=0..1);
C := 13
> A1:=int(5*x+7*y+11*z,x=0..1-(1/2)*y-(1/3)*z);
A1 := 5/2*(1-1/2*y-1/3*z)^2+7*y*(1-1/2*y-1/3*z)+11*z*(1-1/2*y-1/3*z)
> B1:=int(A1,y=0..2*(1-(1/3)*z));
B1 := -23/24*(2-2/3*z)^3+1/2*(-7*z+9/2)*(2-2/3*z)^2+5/2*(1-1/3*z)^2*(2-2/3*z)+11*z*(1-1/3*z)*(2-2/3*z)
> C1:=int(B1,z=0..3);
C1 := 13
```

The `B1` value was centered so nothing got "lost" over the page's edge. From all this we learn that **13=13**.