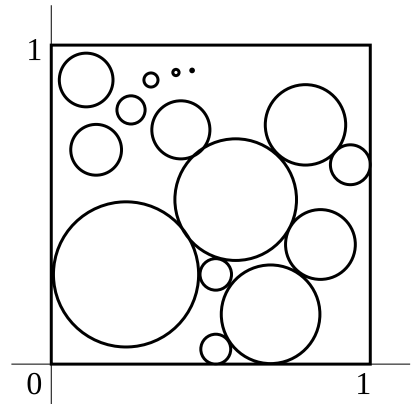
Disconcerting problems about dimensions

Discussion and statement of the first problem

A *sequence of bubbles* is an infinite sequence of circles in the unit square of the plane, $[0,1] \times [0,1]$, whose interiors do not overlap. The center and radius of each circle should be specified in some algebraic or geometric fashion. A picture of some bubbles in one sequence appears to the right.



Is there a sequence of bubbles so that

i the sum of the bubble areas is finite and

ii the sum of the bubble circumferences is infinite?

What you should do

Either give an example of such a sequence of bubbles as explicitly as you can, or explain why no example exists. Your answer should contain a discussion supporting your assertion written in complete English sentences.

Discussion and statement of the second problem

We begin with some terminology and notation.

• $\mathbb{R}^n$ (pronounced "are en") is n-dimensional Euclidean space. A point p in $\mathbb{R}^n$ is an n-tuple of real numbers: $p = (x_1, x_2, \ldots, x_n)$. The numbers x_j are called the coordinates of p. For example, $(1, 2, -3.8, 400, 5\pi)$ is a point in $\mathbb{R}^5$.

• If $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$ are two points in $\mathbb{R}^n$, the distance from p to q is defined to be $D(p,q) = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$. This is supposed to be a natural generalization of the usual formulas for distance in $\mathbb{R}^2$ and $\mathbb{R}^3$: a repetition n times of the Pythagorean formula. For example, $p = (1, 7, 8, -4)$ and $q = (2, -3, 9, 9)$ are points in $\mathbb{R}^4$, and the distance between them is $\sqrt{(1-2)^2 + (7-(-3))^2 + (8-9)^2 + (-4-9)^2} = \sqrt{271} \approx 16.46208$. The formula for $D(p,q)$ satisfies the usual rules for distances. The text uses $|pq|$ to denote the distance from p to q.

• The origin in $\mathbb{R}^n$ is $\mathbf{0} = (0, 0, \ldots, 0)$, the n-tuple which is all 0's.

• The n-dimensional unit cube is the collection of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying all of these inequalities: $0 \le x_j \le 1$ for $1 \le j \le n$.

• The corners of the n-dimensional unit cube are the points $(x_1, x_2, \ldots, x_n)$ where each x_j is either 0 or 1. Each of the n choices of the coordinates for a corner can be made independently and there are two alternatives for each coordinate. Therefore the n-dimensional unit cube has 2^n corners.

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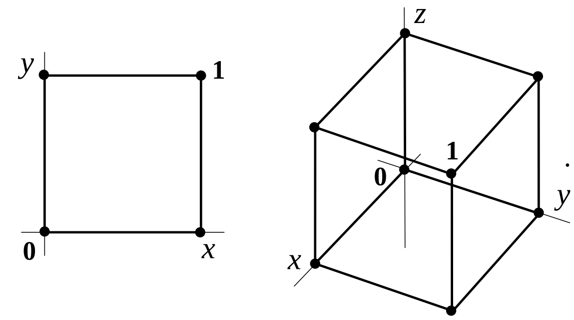

Here are some familiar unit cubes, in 2 and 3 dimensions. The corners are marked with $\bullet$'s. The 2-dimensional cube has $2^2 = 4$ corners. The 3-dimensional cube has $2^3 = 8$ corners.

> Do these exercises before starting the problem. The solutions should *not* be handed in! Bare answers ("spoilers") without explanation appear at the bottom of the page. I suggest you look at them *after* you try the problems.

Exercise 1 Suppose $\mathbf{1} = (1, 1, \ldots, 1)$, the n-tuple which is all 1's. Compute the distance between $\mathbf{0}$ and $\mathbf{1}$, which are both corners of the n-dimensional cube. This should convince you that at least *part* of the n-dimensional cube "sticks out" far away from the origin.

Exercise 2 The 20-dimensional unit cube has $2^{20} = 1{,}048{,}576$ corners, far too many to list explicitly. You may need to use a calculator to answer the questions below.

a) How many corners of the 20-dimensional cube have *all* 0's in their coordinates? How many have *exactly one* 1 in their coordinates? How many have *exactly two* 1's in their coordinates? How many have *exactly three* 1's in their coordinates? How many have *exactly four* 1's in their coordinates? [This starts out very easy, then becomes harder.]

b) Use a)'s answer to find the total number of corners of the 20-dimensional unit cube which have 1's in <u>at most</u> four coordinates.

c) Use b)'s answer to find the total number of corners of the 20-dimensional unit cube whose distance to $\mathbf{0}$ is at most 2. ($2 = \sqrt{1^2 + 1^2 + 1^2 + 1^2}$.)

d) Use c)'s answer to find the <u>proportion</u> of the corners of the 20-dimensional unit cube which have distance to the origin <u>greater than</u> 2.

You may now believe unit cubes are quite weird when n is large. This is true:

> Suppose A is a positive constant. Define $\#(n, A)$ to be the number of corners of the n-dimensional unit cube whose distance to $\mathbf{0}$ is greater than A. Then
> $$\lim_{n\to\infty} \frac{\#(n, A)}{2^n} = 1$$
> so "almost all" of the corners of the cube are eventually, as dimension grows, farther away from $\mathbf{0}$ than A.

What you should do

Verify the limit statement above. (You might want to actually prove that the proportion of corners whose distance is less than A has limit 0.) Use facts from calculus (quote them) comparing the asymptotic growth of polynomials and exponentials. Your answer should have a discussion supporting your assertion written in complete English sentences.

Hints

Begin with $A = 2$: in exercise 2, generalize to $\mathbb{R}^n$ in place of $\mathbb{R}^{20}$. The limit for $A = 2$ compares the growth of a fourth degree polynomial with that of an exponential function. Then consider $A = 78$. The polynomial's degree is now 78^2 but the asymptotics (polynomial growth versus exponential growth) remain qualitatively the same.

Hand in a report on the general case, if possible. The course web page has links which may be useful about binomial coefficients and the symbols used for summation and product.

Answers to exercises: $\sqrt{n}$ | 1, 20, 190, 1140, 4845 | 6196 | 6196 again. | $\approx .99409$