Please write solutions to $N+1$ of these problems by Thursday, October 12. These written solutions should be accompanied by explanations using complete English sentences. Any computer assistance should be appropriately documented. Students may work in groups: a group is N^* students. All students in each group should read all of the group's answers before the work is handed in. Thus all students in each group will therefore be responsible for all answers handed in by the group.

1. a) Compute the curvature of the plane curve $\begin{cases} x(t) = \dfrac{1-t^2}{1+t^2} \\ y(t) = \dfrac{2t}{1+t^2} \end{cases}$. Explain the result using reasoning independent of the curvature computation.

b) Compute the curvature of the space curve $\begin{cases} x(t) = \dfrac{2-t^2}{1+t^2} \\ y(t) = \dfrac{2t^2-2}{1+t^2} \\ z(t) = \dfrac{3t^2-2}{1+t^2} \end{cases}$. Explain the result using reasoning independent of the curvature computation.

Comment/hint I could do the computation in a) "by hand". Probably I could do b) also, but *I would rather get help from a silicon friend*. The cryptic sentences beginning "Explain the result …" mean that after direct computation of the curvatures, there are ways of verifying the results *independently* of the curvature computations.

2. Suppose $y = f(x)$ is a function with domain <u>all of $\mathbb{R}$</u>. Show that the following is *impossible*:

> The curvature $\kappa(x)$ at every point of the graph of $y = f(x)$ is at least 1: $\kappa(x) \geq 1$ for all x.

Comment/hint You can understand this statement geometrically (remember, $f(x) = x^2$ flattens out towards the edges as $x \to \pm\infty$ so it doesn't violate the impossibility assertion!), but I want a verification using calculus. One way is to integrate an inequality.

3. Suppose the curve C has position vector $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}$.

a) What is the unit tangent vector to C when $t = \pi$?

b) Compute the length of the part of C between $t = 0$ and $t = 2\pi$.

c) Verify that the curvature of this curve is given by the formula $\kappa(t) = \sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$. Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty} \kappa(t) = 0$. Explain briefly why this can happen, using complete English sentences possibly assisted by properly labelled diagrams.

Comment/hint I think I would do the computation in a) "by hand". Probably I could do b) and c) that way also, but *I would rather get help from a silicon friend*.

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* N here is 1 or 2 or 3.

4. a) Suppose $h(x,y) = x^y$. Rewrite h as a composition of standard functions, and then find the domain of h and the first partial derivatives of h based on the standard restrictions. I know that $2^3 = 8$. If x is increased by .01 (so (x,y) changes from $(2,3)$ to $(2.01,3)$), approximate the change in h using the linear approximation. If y is increased by .01 (so (x,y) changes from $(2,3)$ to $(2,3.01)$), approximate the change in h using the linear approximation. Compare the "exact answers" to the linearization answers.

b) Suppose $j(x,y,z) = x^{(y^z)}$. Rewrite j as a composition of standard functions, and then find the domain of j and the first partial derivatives of j based on the standard restrictions. I know that $2^{(3^4)} \approx 2.417 \cdot 10^{24}$*. If one of the variables is increased by .01, which variable will likely make the biggest change in the value of j? Support your assertion by an argument using linear approximation based on the derivatives which have been calculated**.

5. Suppose $R(x,y) = v(x+y^2)$ where v is a four times differentiable function of *one* variable. Suppose you know also that:

$$v(0) = \alpha,\ v'(0) = \beta,\ v^{(2)}(0) = \gamma,\ v^{(3)}(0) = \delta,\ v^{(4)}(0) = \epsilon$$

Compute the seven quantities:

$$R(0,0),\ \frac{\partial R}{\partial x}(0,0),\ \frac{\partial R}{\partial y}(0,0),\ \frac{\partial^2 R}{\partial x^2}(0,0),\ \frac{\partial^2 R}{\partial y^2}(0,0),\ \frac{\partial^2 R}{\partial x \partial y}(0,0),\ \frac{\partial^4 R}{\partial x^2 \partial y^2}(0,0)$$

in terms of α, β, γ, δ, and ϵ.

6. Suppose $g(t) = Q(t^3, t^5)$. Suppose you also know that

$$Q(1,1) = A,\ \frac{\partial Q}{\partial x}(1,1) = B,\ \frac{\partial Q}{\partial y}(1,1) = C,\ \frac{\partial^2 Q}{\partial x^2}(1,1) = D,\ \frac{\partial^2 Q}{\partial x \partial y}(1,1) = E,\ \frac{\partial^2 Q}{\partial y^2}(1,1) = F$$

Compute the quantities $g(1)$, $g'(1)$, and $g''(1)$ in terms of A, B, C, D, and E.

7. A rectangular box with an open top has a square base. The sides are made of cardboard, costing 3 cents per square foot. The base is made of plywood, costing a half dollar per square foot. The box should have a capacity of no more than 10 cubic feet and no less than 2 cubic feet. At the same time, due to limitations of construction, no edge of the box should be shorter than 3 inches or longer than 36 inches. Find a plausible domain for the dimensions of the box based on these specifications and describe the domain carefully, algebraically. Sketch the domain in $\mathbb{R}^2$. (You *must* give a complete algebraic description of the domain, however. The picture is *not* a substitute for this description.) Write a formula for a function which calculates the cost of the materials in each possible box.

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* Actually, it is *exactly* 24178 51639 22925 83494 12352.

** Please note that I am *not* interested in the exact values of the perturbed function here, only in the linearized approximations to the perturbed function values.