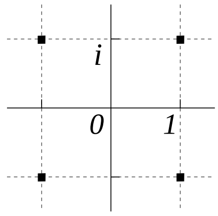
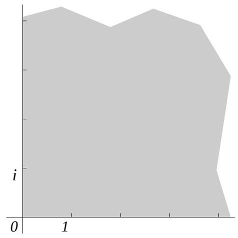
640:403:02 **Answers to the First Exam** **3/4/2005**



(10) 1. Describe all solutions of $z^4 = -4$ algebraically in rectangular form. Sketch the solutions on the axes provided.

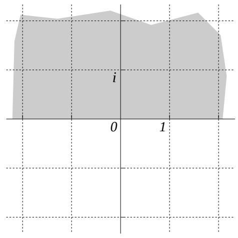
Answer If $-4 = 4e^{i\pi}$, so one fourth root is $\sqrt{2}e^{i\frac{\pi}{4}}$. This is $\sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right) = 1+i$. The other fourth roots are rotated by $\frac{\pi}{2}$, and so they are $1+i$, $-1+i$, $-1-i$, and $1-i$. The roots are indicated by the (square) dots in the graph.



(14) 2. In this problem the open first quadrant of $\mathbb{C}$ will be denoted by Q. A number z is in Q if both $\operatorname{Re} z > 0$ and $\operatorname{Im} z > 0$. A portion of Q is shown to the right.

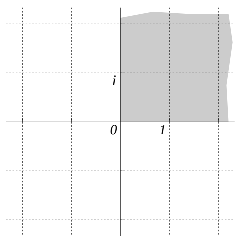
a) Write the complex polar expression for z and indicate what restrictions on r and θ are needed so that z is guaranteed to be in Q.

Answer Here $z = re^{i\theta}$ and $r > 0$ and $0 < \theta < \frac{\pi}{2}$.



b) What is the image of Q under the mapping $z \mapsto z^2$? Indicate the portion of the plane which is the image on the axes to the right and briefly explain your drawing algebraically.

Answer Since $z^2 = r^2e^{2i\theta}$, we see that the modulus is still any positive number, but the argument is between 0 and π. This is the open upper half plane.



c) What is the image of Q under the mapping $z \mapsto \frac{i}{z}$? Indicate the portion of the plane which is the image on the axes to the right and briefly explain your drawing algebraically.

Answer Now $\frac{i}{z} = \frac{e^{i\frac{\pi}{2}}}{re^{i\theta}} = \frac{1}{r}e^{i\left(\frac{\pi}{2}-\theta\right)}$. Since $0 < r < \infty$, $\frac{1}{r}$ will describe the same interval. The argument, $\frac{\pi}{2} - \theta$, is the interval from 0 to $\frac{\pi}{2}$. The result is the open first quadrant.

d) Verify the following assertion: if z is in Q, then $\frac{z^3+i}{z}$ is in the open upper half plane (those complex numbers whose imaginary part is positive). You may use the conclusions of parts b) and c) combined with geometric reasoning, or some other method.

Answer Notice that $\frac{z^3+i}{z} = z^2 + \frac{i}{z}$. Therefore we are dealing with the sum of two complex numbers, one which is in the open upper half plane and the other which is in the open first quadrant. The sum of two vectors whose tail is at the origin and which point *up* also points *up*. More prosaically, if $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ with $b_1 > 0$ and $b_2 > 0$, the sum $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$ also has imaginary part $b_1 + b_2$ greater than 0.

(12) 3. Suppose that $f(z)$ is an analytic function with real part $u(x,y)$ and imaginary part $v(x,y)$, and that $u(x,y) = \big(v(x,y)\big)^2$ always. Show that $f(z)$ must be constant. **Hint** The Cauchy-Riemann equations.

Answer if $u = v^2$ then $u_x = 2vv_x$. The Cauchy-Riemann equations imply that $u_x = 2v(-u_y)$. Similarly, $u_y = 2vv_y = 2vu_x$. Substitute the second equation into the first and get $u_x = 2v(-2vu_x) = -4v^2u_x$. Therefore, $(1 + 4v^2)u_x = 0$. The real function $1 + 4v^2$ is never 0, so u_x must always be 0. Take the first equation and substitute into the second: $u_y = 2v(2v\{-u_y\}) = -4v^2u_y$. Just as before, we see that u_y must always be 0. We proved that in a connected open set (such as the domain of an analytic function), a function whose partial derivatives are 0 must be constant. Thus u is constant. The Cauchy-Riemann equations imply that the partial derivatives of v are always 0, so that v is constant. Therefore $f(z)$ is constant.

(10) 4. Find all values of A so that $e^{3x}\sin(Ay)$ is harmonic, and find a harmonic conjugate for each of these harmonic functions.

Answer $\frac{\partial^2}{\partial x^2}\left(e^{3x}\sin(Ay)\right) + \frac{\partial^2}{\partial y^2}\left(e^{3x}\sin(Ay)\right) = 9e^{3x}\sin(Ay) - A^2e^{3x}\sin(Ay)$. This is 0 when A^2 is 9, so $A = \pm 3$. To find a harmonic conjugate: we know that $u_x = 3e^{3x}\sin(Ay) = v_y$. We can antidifferentiate this and get $v = -\frac{3}{A}e^{3x}\cos(Ay) + \text{STUFF}_1(y)$. Also, $-u_y = -(Ae^{3x}\cos(Ay) = v_x$ and therefore $v = -\frac{A}{3}e^{3x}\cos(Ay) + \text{STUFF}_2(x)$. So one harmonic conjugate when u is $e^{3x}\sin(\pm 3y)$ is $v = \mp e^{3x}\cos(\pm 3y)$.

Comment $A = 0$ is also a valid answer which several students found but the instructor did *not*! His work was also *not* perfect! The question should have asked about $e^{Ax}\sin(3y)$ so this difficulty wouldn't occur.

(12) 5. a) Find $\log(-1)$ and $(-1)^i$.

Answer $\log(-1) = \ln(|-1|) + i\arg(-1) = 0 + i(\pi + 2n\pi)$ for any integer n. Then $(-1)^i = e^{i(\log(-1))} = e^{i^2(\pi+2n\pi} = e^{-\pi+2n\pi}$.

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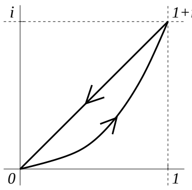
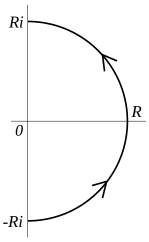
b) Provide an explicit pair of values z and w so that $\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$.
Answer There are many pairs of z and w which will provide valid answers. If $z = w = -1+i$, then $zw = -2i$. $\operatorname{Log}(-2i) = \ln 2 - i\pi$ and $\operatorname{Log}(-1+i) = \ln\sqrt{2} + \frac{3}{4}\pi i$, so that $\operatorname{Log} z + \operatorname{Log} w = 2\left(\ln\sqrt{2} + \frac{3}{4}\pi i\right) = \ln 2 + \frac{3}{2}\pi i$, which is *not* the same as $\ln 2 - i\pi$.

(10) 6. Define $\sin z$ and $\cos z$ in some way appropriate for this course, and use your definitions to prove that $(\sin z)^2 + (\cos z)^2 = 1$ for all complex z.
Answer Here are good definitions: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$. So $(\sin z)^2 = \frac{e^{2iz} - 2e^{iz}e^{-iz} + e^{-2iz}}{(2i)^2} = \frac{e^{2iz} - 2 + e^{-2iz}}{-4} = \frac{-e^{2iz} + 2 - e^{-2iz}}{4}$ and $(\cos z)^2 = \frac{e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}}{2^2} = \frac{e^{2iz} + 2 + e^{-2iz}}{4}$. Therefore $(\sin z)^2 + (\cos z)^2 = \frac{-e^{2iz} + 2 - e^{-2iz}}{4} + \frac{e^{2iz} + 2 + e^{-2iz}}{4} = \frac{4}{4} = 1$.
Other acceptable definitions use the formulas for sine and cosine on p. 50 of the text. It is then possible, using "known facts" about the trig and hyperbolic functions, to deduce the identity $(\sin z)^2 + (\cos z)^2 = 1$ for all complex z. It *is* possible, but the process is longer than what I've already written.



(12) 7. Compute $\int_C x^2\,dz$ where C is the simple closed curve obtained by following the parabola $y = x^2$ from 0 to $1+i$, and then following the line $y = x$ from $1+i$ to 0. The curve is displayed to the right.
Answer I believe that the easiest way to do this problem is to parameterize the two pieces of the curve and compute. Then the parabolic arc becomes: $z = t + it^2$ with $0 \le t \le 1$ so $dz = (1 + 2it)dt$ and $x^2 = t^2$. The integral is $\int_{\text{PARABOLIC ARC}} x^2\,dz = \int_0^1 t^2(1+2it)dt = \int_0^1 t^2 + 2it^3\,dt = \frac{1}{3}t^3 + \frac{2i}{4}t^4\Big]_{t=0}^{t=1} = \frac{1}{3} + \frac{i}{2}$. The line segment is parameterized by $z = t + it$ so $dz = (1+i)dt$ and $x^2 = t^2$. The line segment is reversed, though (look at the picture!), so the endpoints of the integral will go from 1 to 0. This piece of the line integral is $\int_{\text{REVERSED LINE SEGMENT}} x^2\,dz = \int_1^0 t^2(1+i)dt = \int_1^0 t^2 + it^2\,dt = \frac{1}{3}t^3 + \frac{i}{3}t^3\Big]_1^0 = -\frac{1}{3} - \frac{i}{3}$. The total is therefore $\frac{1}{3} + \frac{i}{2} - \frac{1}{3} - \frac{i}{3} = \frac{i}{6}$.
What about Green's Theorem? I get the signs wrong frequently. The line integral is equal to a double integral over the region bounded by the line segment and parabolic arc. The integrand is $i(f_x + if_y)$ which in our case is $i(2x)$. We need the boundaries of the double integral: x goes from 0 to 1, and y, the "inside" variable, will go from x^2 to x. Therefore the double integral we need to compute is $\int_0^1 \int_{x^2}^x 2ix\,dy\,dx$. First let's do the inner integral: $\int_{x^2}^x 2ix\,dy = 2ixy\big]_{x^2}^x = 2ix^2 - 2ix^3$. Then the outer integral becomes $\int_0^1 2ix^2 - 2ix^3\,dx = \frac{2i}{3}x^3 - \frac{2i}{4}x^4\Big]_0^1 = \frac{2i}{3} - \frac{2i}{4} = \frac{i}{6}$. Of course, doing the double integral as $\int_0^1 \int_y^{\sqrt{y}} 2ix\,dx\,dy$ gives the same answer.



(12) 8. Verify that $\lim_{R\to\infty} \int_{C_R} \frac{e^{-z} + 2z}{z^3 + 6z^2}\,dz = 0$ where C_R is the semicircle of radius R and center 0 in the right half plane going from $-Ri$ to Ri as shown. Show details of your estimates.
Answer We will use the ML inequality to estimate the modulus of the integral. The length is πR. M is more difficult. The top of the integrand is $e^{-z} + 2z$, and certainly $|e^{-z} + 2z| \overset{\triangle}{\le} |e^{-z}| + |2z|$. But $|e^{\text{FROG}}| = e^{\operatorname{Re}\text{FROG}}$. Since z is in the right half plane, the real part of $-z$ is at most 0. Also $|2z| = 2R$ on C_R. Therefore the top is bounded by $1 + 2R$. We need an underestimate of the bottom. So $|z^3 + 6z^2| \underset{\nabla}{\ge} |z^3| - |6z^2| = |z|^3 - 6|z|^2$. Again, on C_R, $|z| = R$ so that $|z^3 - 6z^2| \ge R^3 - 6R^2$ which is positive for large R. Therefore M is at most $\frac{1+2R}{R^3 - 6R^2}$. The modulus of the integral thus is bounded by $ML = \frac{(\pi R)(1+2R)}{R^3 - 6R^2}$. Since this is a rational function whose top degree is 2 and whose bottom degree is 3, and $3 > 2$, the limit as $R \to \infty$ is 0 as desired.

(8) 9. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n(n+1)}{(3+4i)^n} z^n$.
Answer Inside the radius of convergence the series will converge absolutely, and outside that radius, it will diverge. Therefore we need only consider absolute convergence. I'll use the Ratio Test. Here $a_n = \frac{n(n+1)}{(3+4i)^n} z^n$ so that $a_{n+1} = \frac{(n+1)(n+2)}{(3+4i)^{n+1}} z^{n+1}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(n+1)(n+2)}{(3+4i)^{n+1}} z^{n+1}}{\frac{n(n+1)}{(3+4i)^n} z^n}\right| = \left|\frac{(n+1)(n+2)z^{n+1}(3+4i)^n}{n(n+1)z^n(3+4i)^{n+1}}\right| = \left|\frac{(n+2)z}{n(3+4i)}\right| = \left(\frac{n+2}{n}\right)\frac{|z|}{|3+4i|} = \left(\frac{n+2}{n}\right)\frac{|z|}{5}$. As $n \to \infty$, the limit is $\frac{|z|}{5}$. Convergence will occur when this limit is less than 1, and divergence, when it is greater than 1. Therefore the radius of convergence is 5.