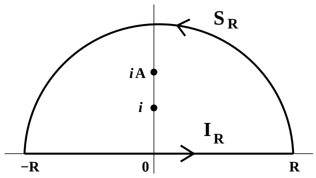
The symbol "…" is used to mean "higher order terms" in the solutions.

(15) 1. If A is real and positive, and $A \neq 1$, `Maple` reports that $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+A^2)} = \frac{\pi}{A+1}$. Check this assertion using the Residue Theorem. **Sketch** the contour of integration. Show any residue computations. Explain why some integral has limiting value equal to 0.

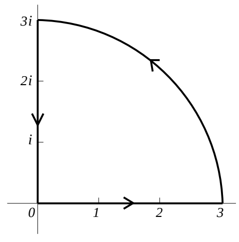


Answer Suppose R is a positive real number, and $R > \max(1, A)$. We will use the contour $C_R = I_R + S_R$, oriented as shown, where I_R is the interval from $-R$ to R on $\mathbb{R}$ and S_R is the semicircle in the upper half plane, centered at 0 with radius R. Also let $f(z) = \frac{z^2}{(z^2+1)(z^2+A^2)}$, analytic except for isolated singularities at $\pm i$ and $\pm A$. Since $(z^2+1)(z^2+A^2) = (z+i)(z-i)(z-iA)(z+iA)$, and $z^2 \neq 0$ at $\pm i$ and $\pm A$, each of the singularities is a simple pole. The residue of $f(z)$ at $z = i$ is therefore $\frac{i^2}{2i(i-iA)(i+iA)} = \frac{-1}{2i(A^2-1)}$, and the residue of $f(z)$ at $z = iA$ is therefore $\frac{(iA)^2}{(iA+i)(iA-i)(2iA)} = \frac{A}{2i(A^2-1)}$. The sum of these two residues is $\frac{-1+A}{2i(A^2-1)} = \frac{1}{2i(A+1)}$. The Residue Theorem then declares that $\int_{C_R} f(z)\,dz = 2\pi i\left(\frac{1}{2i(A+1)}\right) = \frac{\pi}{A+1}$. Now as $R \to +\infty$, $\int_{I_R} f(z)\,dz \to \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+A^2)}$. What about $\int_{S_R} f(z)\,dz$? On S_R, $|z| = R > \max(1, A)$ so that $|f(z)| \leq \left|\frac{z^2}{(z^2+1)(z^2+A^2)}\right| = \frac{|z|^2}{(|z|^2-1)(|z|^2-A^2)} \leq \frac{R^2}{(R^2-1)(R^2-A^2)}$. So ($ML$ as usual), $\left|\int_{S_R} f(z)\,dz\right| \leq \frac{2\pi R^3}{(R^2-1)(R^2-A^2)}$. Since $4 > 3$, this estimate shows that $\int_{S_R} f(z)\,dz \to 0$ as $R \to +\infty$. Therefore the real integral we want does turn out to be the quantity `Maple` computed.

For a rather small amount of extra credit* please *guess* the value of $\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2}$.
Answer $\frac{\pi}{2}$. (Let $A \to 1$ in both the integral and the answer.)

(14) 2. a) State a version of the Cauchy integral formula for derivatives**.
Answer Suppose that $f(z)$ is analytic in a simply connected domain, D, that C is a positively oriented simple closed curve in D, that z_0 is a point inside C, and that n is a non-negative integer. Then $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$.

b) Use the statement in a) or some other result to compute $\int_B \frac{z^4}{\left(z-(1+i)\right)^3}\,dz$ where B is the simple closed curve shown: the line segment from 0 to 3, followed by the quarter-circular arc centered at 0 from 3 to $3i$, followed by the line segment from $3i$ to 0. **Answer** -24π



Answer Here $f(z) = z^4$, $n = 2$, $z_0 = 1+i$, and C is the curve B. Then $f^{(2)}(1+i) = 4\cdot 3(1+i)^2 = 24i$. The answer we want is $24i$ divided by $2! = 2$ and multiplied by $2\pi i$, so the answer is -24π. The answer can also be obtained using the Residue Theorem, or by computing the appropriate Laurent series and integrating directly.

(15) 3. In this problem, $f(z) = \frac{1}{e^z-1} - \frac{1}{z}$. Find and classify (as removable, pole, or essential) all isolated singularities of $f(z)$. Find the residue of $f(z)$ at *every* singularity. For each singularity which is a pole, find the order of the pole. **Comment** This is a *complex* analysis course! Please find **all** isolated singularities!
Answer Certainly 0 is an isolated singularity of $f(z)$. This is because both $e^z - 1$ and z are 0 at $z = 0$. Certainly $\frac{1}{z}$ has no other singularities, but $\frac{1}{e^z-1}$ has many. In fact, $e^z = 1$ when $z = 2\pi n i$ for any integer n.

$\boxed{n = 0}$ For $z \neq 0$, $\frac{1}{e^z-1} = \frac{1}{z+\frac{z^2}{2}+\ldots} = \frac{1}{z}\left(\frac{1}{1+\frac{z}{2}+\ldots}\right) = \frac{1}{z}\left(1 - \frac{z}{2} + \ldots\right)$ so that $f(z) = \frac{1}{z}\left(1 - \frac{z}{2} + \ldots\right) - \frac{1}{z} = \frac{1}{z}\left(1 - \frac{z}{2} + \ldots - 1\right) = -\frac{1}{2} + \ldots$. Therefore $f(z)$ has a removable singularity at $z = 0$. The residue of $f(z)$ at $z = 0$ is 0.

$\boxed{n \neq 0}$ I know that $e^z = 1 + (z-2\pi i n)^1 + \frac{(z-2\pi i n)^2}{2!} + \ldots$ so for $z \neq 2\pi i n$, $f(z) = \frac{1}{e^z-1} - \frac{1}{z} = \frac{1}{z-2\pi i n}(1+\ldots) - \frac{1}{z}$. Thus $f(z)$ is a difference of a function which has a simple pole at $2\pi i n$ and one which is analytic near $2\pi i n$ (for $n \neq 0$). Therefore $f(z)$ has a simple pole at $2\pi i n$ and its residue there is 1.

OVER

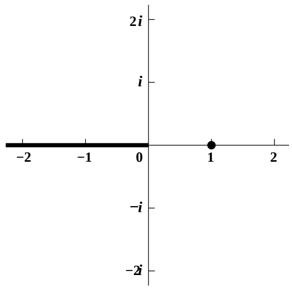
* Let's say, uhhh, 3 points.

** Your statement should contain the words "simply" and "simple", and be valid for derivatives of any order.

(12) 4. a) Suppose that $f(z)$ is an entire function and there is a positive constant K so that $|f(z)| > K$ for all z. Prove that f must be a constant function. **Hint** What can you do with something that is *not* 0?
Answer $f(z)$ can never be 0 since $|f(z)|$ is always positive. Therefore the function $g(z)$ defined by the formula $g(z) = \frac{1}{f(z)}$ is defined for all z in $\mathbb{C}$ and is analytic: $g(z)$ is entire. Also, $|g(z)| < \frac{1}{K}$ for all z, so $g(z)$ is *bounded and entire.* Liouville's Theorem applies to show that $g(z)$ is constant and therefore so is $f(z)$.
b) The exponential function is never 0 and is an entire function. Briefly explain why the exponential function does not contradict the assertion in part a). **Answer** Every non-zero complex (and therefore real) number is a value of the exponential function (which is why log has values at every non-zero complex number). If $K > 0$, there is z_0 with $e^{z_0} = \frac{K}{2}$ so any inequality of the form $|e^z| > K$ is false for some z's.

(14) 5. Find the first four non-zero terms of the Taylor series centered at $z = 0$ for the function $f(z) = \frac{\sin z}{(z-1)^2}$.
Suggestion A direct computation *is* possible. The results are messy, with a large chance for error. Try another way. **Answer** Surely $\sin z = z - \frac{z^3}{3!} + \ldots$ and also ($\frac{a}{1-r}$ again) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots$. Therefore $\frac{1}{(1-z)^2} = (1 + z + z^2 + z^3 + \ldots)^2 = 1 + 2z + 3z^2 + 4z^3 + \ldots$. You can also get this result by differentiating the series for $\frac{1}{1-z}$. Then the series for $f(z)$ is the product: $\left(z - \frac{z^3}{3!} + \ldots\right)\left(1 + 2z + 3z^2 + 4z^3 + \ldots\right)$. We can compute this, and get $z + 2z^2 + \left(-\frac{1}{6} + 3\right) z^3 + \left(4 - \frac{2}{6}\right) z^4 + \ldots = z + 2z^2 + \frac{17}{6}z^3 + \frac{11}{3}z^4 + \ldots$.

(15) 6. In this problem, $f(z) = \frac{\sqrt{z}}{(z-1)^2}$. a) Specify precisely a maximal (largest) domain in $\mathbb{C}$ in which $f(z)$ is analytic. (You need *not* prove your assertion!) **Answer** Certainly we must exclude 1, since otherwise we'll divide by 0. And we need a good domain for $\sqrt{z}$: take all of $\mathbb{C}$ except for real z's with $z \le 0$. So one possible domain is: all z's except for $z = 1$ and real z's with $z \le 0$.



b) What is the radius of convergence of the Taylor series centered at $z = 1 + i$ for $f(z)$? **Note** The coefficients of this series are complicated. You may use your answer to a) here. **Answer** The closest singularity of $f(z)$ is at $z = 1$: the radius of convergence is 1.
c) What is the radius of convergence of the Taylor series centered at $z = -2+i$ for $f(z)$? **Note** The coefficients of this series are complicated. The answer is *tricky.* **Answer** $f(z)$ cannot be continued analytically beyond $z = 0$, since there can be no analytic $\sqrt{z}$ around 0. The radius of convergence is $\le \sqrt{5}$, the distance to 0. The radius of convergence actually *is* $\sqrt{5}$ since we can define $\sqrt{z}$ in any disc not containing 0.
d) Find the first four non-zero terms of the Laurent series centered at $z = 1$ for $f(z)$.
Answer Suppose $g(z)$ is the "principal branch" of $\sqrt{z}$. Then $g(z)$ agrees with the usual (calculus) square root on the positive reals. Therefore $g(1) = 1$, $g'(z) = \frac{1}{2}z^{-\frac{1}{2}}$ so $g'(1) = \frac{1}{2}$, $g''(z) = -\frac{1}{4}z^{-\frac{3}{2}}$ so $g''(1) = -\frac{1}{4}$, and $g'''(z) = \frac{3}{8}z^{-\frac{5}{2}}$ so $g'''(1) = \frac{3}{8}$. The first four non-zero terms of the Taylor series for $g(z)$ centered at $z = 1$ for $g(z)$ are (don't forget the factorials!) $1 + \frac{1}{2}(z-1)^1 - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3$. Therefore the Laurent series for $f(z)$ begins with $(z-1)^{-2} + \frac{1}{2}(z-1)^{-1} - \frac{1}{8} + \frac{1}{16}(z-1)$.

(15) 7. Suppose a and b are real and positive, and $a > b$. Use the Residue Theorem or other results of this course to compute $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$. **Note** This is problem #7 of section 2.3 in your text. The answer given in the text is $\frac{\pi}{\sqrt{a^2-b^2}}$ which seems to be slightly (?) incorrect. **Answer** Use the standard substitution: $e^{i\theta} = z$ so that $dz = ie^{i\theta}d\theta$, and $\cos\theta = \frac{e^{i\theta}-e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$. Therefore $\int_0^{2\pi}\frac{d\theta}{a+b\cos\theta} = \int_{|z|=1}\frac{1}{a+b\left(\frac{z+\frac{1}{z}}{2}\right)}\frac{dz}{iz} = \frac{2}{i}\int_{|z|=1}\frac{1}{2a+b\left(z+\frac{1}{z}\right)}\frac{dz}{z} = \frac{2}{i}\int_{|z|=1}\frac{dz}{bz^2+2az+b} = \frac{2}{ib}\int_{|z|=1}\frac{dz}{z^2+2\left(\frac{a}{b}\right)z+1}$. Let $c = \frac{a}{b}$. The function $f(z) = \frac{1}{z^2+2cz+1}$ is analytic everywhere except at the roots of the quadratic in the denominator. Those roots are $\frac{-2c\pm\sqrt{4c^2-4}}{2} = -c \pm \sqrt{c^2-1}$. I'll call these roots r_+ and r_-, respectively. So $f(z) = \frac{1}{(z-r_+)(z-r_-)}$. Since $|c| > 1$, r_- is outside $|z| = 1$ and r_+ is inside $|z| = 1$. To apply the Residue Theorem, we need the residue of $f(z)$ at $z = r_+$, and that is $\frac{1}{r_+-r_-} = \frac{1}{2\sqrt{c^2-1}}$. Thus the integral is $(2\pi i)\left(\frac{2}{ib}\right)\frac{1}{2\sqrt{c^2-1}} = \frac{2\pi b}{\sqrt{\left(\frac{a}{b}\right)^2-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}$, twice the answer given in the text. You can check this by letting $b \to 0^+$. The integrand becomes close to the constant $\frac{1}{a}$ so the integral should be close to $\frac{2\pi}{a}$. I remark that if you don't "factor out" the b multiplying z^2 you need to compensate carefully when you find the residue. *Monic* polynomials (with leading coefficient equal to 1) are easiest to deal with.