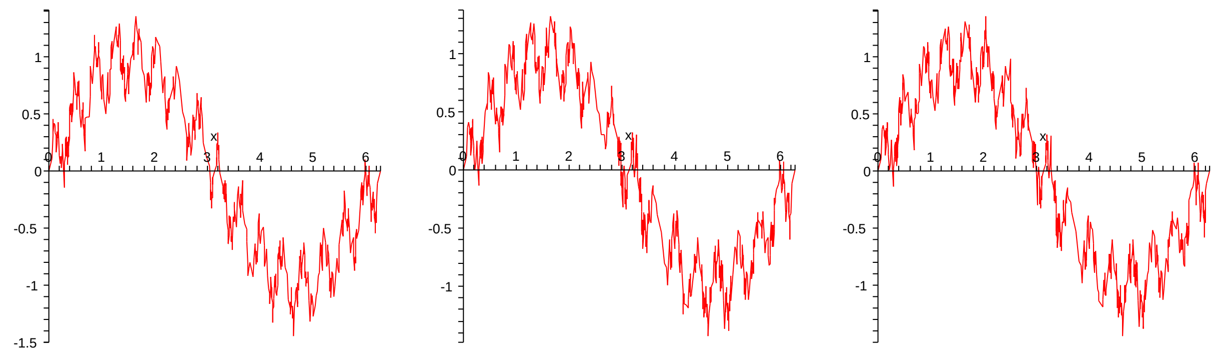
Math 403, section 2 — **A Fourier series example** — February 24, 2005

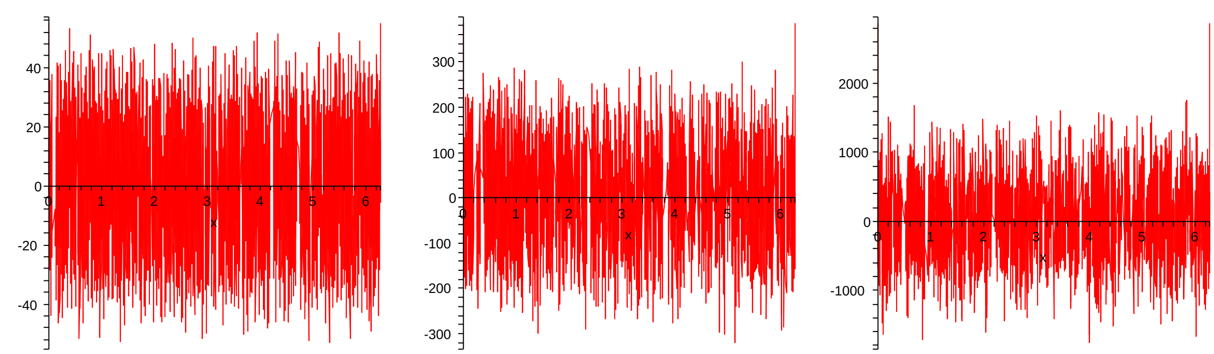
A Fourier series example

Just as power series are infinite linear combinations of monomials, $\{x^n\}_{n=0}^{\infty}$, Fourier series (the specific series given here is called a Fourier sine series) are infinite linear combinations of the functions $\{\sin nx\}_{n=0}^{\infty}$. Power series have very nice behavior inside their radius of convergence: "The sum is continuous inside the radius of convergence" and "The series can be differentiated 'term-by-term' inside the radius of convergence". Fourier series generally aren't as nice as power series, and this example is presented to explain why these properties of power series should *not* be accepted as CLEAR.

The example is the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(n^4 x\right)}{n^2}$ on the interval $[0, 2\pi]$. Since all the sines are 2π periodic, this interval displays all of the behavior of the sum. Suppose N is a positive integer and F_N is defined by $F_N(x) = \displaystyle\sum_{n=1}^{N} \frac{\sin\left(n^4 x\right)}{n^2}$, the N^{th} partial sum of the infinite series. From left to right below are the graphs of F_5, F_{10}, and F_{20} as drawn by `Maple`.



It certainly *seems* reasonable that these pictures are stabilizing, and tending to some complicated but continuous limit. Now here are `Maple` pictures of the derivatives of F_5, F_{10}, and F_{20}.



Look at the the vertical axes. The derivatives are behaving even more wildly than a first glance would indicate. They are growing and wiggling enormously. Nothing is canceling. It seems reasonable that the sum of the derivatives would *not* converge.

$\Longrightarrow$ **Power series behave very nicely; Fourier series may not.** $\Longleftarrow$