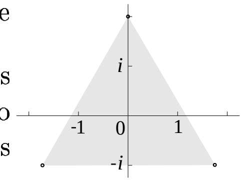
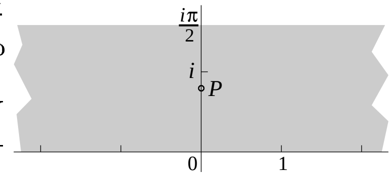
(9) 1. Describe all solutions of $z^3 = -8i$ algebraically in rectangular form. Sketch the solutions on the axes provided.

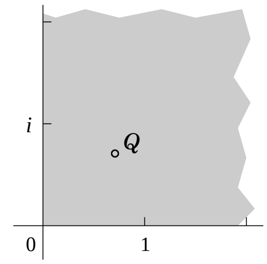


Answer Since $-8i = 8e^{i3\pi/2}$, one cube root is $8^{1/3}e^{i\frac{\pi}{2}}$ which is $2i$. The other cube roots are rotated by an angle of $\frac{2\pi}{3}$. That's multiplication by $w = e^{i2\pi/3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. So another root is $2iw = -\sqrt{3} - i$ and the third root is $w\left(-\sqrt{3} - i\right) = \sqrt{3} - i$. The roots are shown on the graph and form an equilateral triangle.

(14) 2. In this problem U is the open horizontal strip defined by $0 < \operatorname{Im} z < \frac{\pi}{2}$ and P is the point $\frac{i\pi}{4}$ in U. Part of U is shown below, and P is also indicated.

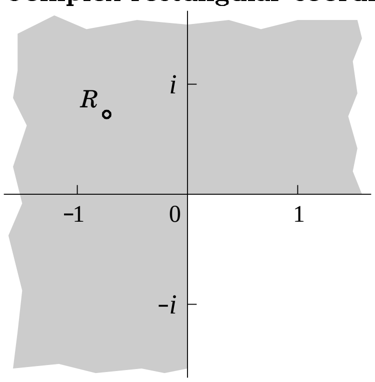


a) Suppose V is the image of U under the exponential mapping. Sketch V <u>on the axes provided to the right</u>. Write an algebraic description of V below. Also label the point Q, the image of P under the exponential mapping, <u>on your sketch</u>, and write rectangular complex coordinates for Q.



Algebraic description of V (Complex polar or rectangular) z is in V when $z = re^{i\theta}$ where $r > 0$ and $0 < \theta < \frac{\pi}{2}$. Alternatively, z is in V when $\operatorname{Im} z > 0$ and $\operatorname{Im} z > 0$.

Complex rectangular coordinates for Q This is $e^{i\pi/4} = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.



b) Suppose that W is the image of V (NOTE: V) under the cubing mapping (that is, $z \to z^3$). Sketch W <u>on the axes provided below</u>. Write an algebraic description of W below. Additionally, label the point R, the image of Q (NOTE: Q) under the cubing mapping, <u>on your sketch</u>, and write rectangular complex coordinates for R.

Algebraic description of W (Complex polar or rectangular) z is in W when $z = re^{i\theta}$ where $r > 0$ and $0 < \theta < \frac{3\pi}{2}$. Alternatively, z is in V when at least one of these inequalities are *false*: $\operatorname{Re} z > 0$ and $\operatorname{Im} z < 0$. So z is in V if $\operatorname{Im} z > 0$ or if $\operatorname{Re} z < 0$ and $\operatorname{Im} z < 0$.

Complex rectangular coordinates for R This is $e^{i3\pi/4} = -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

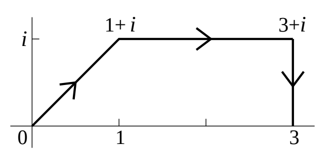
(12) 3. a) Find all values in rectangular complex form of $(1+i)^i$ and $(1+i)^2$.

Answer $a^b = e^{b\log a}$ so $(1+i)^i = \exp\left(i(\ln(\sqrt{2}) + i\frac{\pi}{4} + 2\pi n i\right) = e^{-\frac{\pi}{4}-2\pi n}e^{i\ln(\sqrt{2})} = e^{-\frac{\pi}{4}-2\pi n}\left(\cos(\ln(\sqrt{2})) + i\sin(\ln(\sqrt{2}))\right) = e^{-\frac{\pi}{4}-2\pi n}\cos(\ln(\sqrt{2})) + ie^{-\frac{\pi}{4}-2\pi n}\sin(\ln(\sqrt{2}))$ where n is any integer. Also, $(1+i)^2 = \exp\left(2(\ln(\sqrt{2}) + i\frac{\pi}{4} + 2\pi n i\right) = e^{i\frac{\pi}{2}+4\pi n i}e^{2\ln(\sqrt{2})} = e^{i\frac{\pi}{2}}e^{\ln(2)} = 2i$. There is only one value because the exponential function is periodic with period $2\pi i$.

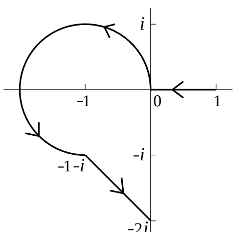
b) Provide an explicit pair of values z and w so that $\operatorname{Log}(zw) \neq \operatorname{Log} z + \operatorname{Log} w$.

Answer There are many pairs of z and w which will provide valid answers. If $z = w = -1+i$, then $zw = -2i$. $\operatorname{Log}(-2i) = \ln 2 - i\pi$ and $\operatorname{Log}(-1+i) = \ln\sqrt{2} + \frac{3}{4}\pi i$, so that $\operatorname{Log} z + \operatorname{Log} w = 2\left(\ln\sqrt{2} + \frac{3}{4}\pi i\right) = \ln 2 + \frac{3}{2}\pi i$, which is *not* the same as $\ln 2 - i\pi$.

(13) 4. a) Compute $\int_C x\,dz$ where C is the curve given by a line segment from 0 to $1+i$ followed by a line segment from $1+i$ to $3+i$, and then followed by a line segment from $3+i$ to 3. This curve is displayed to the right.



Answer C is a sum of three line segment: $C = C_1 + C_2 + C_3$. From 0 to $1+i$, $z = (1+i)t$ so $dz = (1+i)dt$ with $0 \le t \le 1$ and $x = t$: $\int_{C_1} x\,dz = \int_0^1 t(1+i)\,dt = \int_0^1 t(1+i)dt = (1+i)\frac{1}{2}$. From $1+i$ to $3+i$, $z = t+i$ so $dz = dt$ with $1 \le t \le 3$ and $x = t$: $\int_{C_2} x\,dz = \int_1^3 t\,dt = \frac{9}{2} - \frac{1}{2} = 4$. From $3+i$ to 3, $z = 3+ti$ so $dz = i\,dt$ with $1 \ge t \ge 0$ (direction!) and $x = 3$: $\int_{C_3} x\,dz = \int_1^0 3i\,dt = -3i$. Therefore $\int_C x\,dz = (1+i)\frac{1}{2} + 4 - 3i$.



b) Compute $\int_C (z^2+1)\,dz$ where C is the curve given by a line segment from 1 to 0, followed by a circular arc of radius 1 centered at -1, connected 0 to $-1-i$, followed by a line segment from $-1-i$ to $-2i$. This curve is displayed to the right.

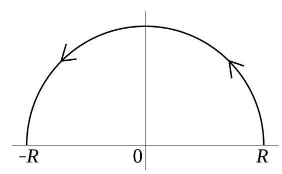
Answer z^2+1 is the derivative of $F(z) = \frac{1}{3}z^3 + z$ and therefore the integral can be evaluated as $F(\text{END}) - F(\text{START}) = \frac{1}{3}(-2i)^3 - 2i - \left(\frac{1}{3}(1)^3 + 1\right)$.

Alternative answer to a) We can use Green's Theorem. Suppose TRAP is the trapezoid whose vertices are 0, $1+i$, $3+i$, and 3, and D is the boundary of TRAP. Then $\int_D x\,dz = \int_D x\,(dx + i\,dy) = \int_D x\,dx + ix\,dy = \int_D P\,dx + Q\,dy = \int\int_{\text{TRAP}} Q_x - P_y\,dx\,dy = \int\int_{\text{TRAP}} i\,dx\,dy = i\left(\frac{5}{2}\right)$. Now C is part of D but reversed, so $C = -D + I$, where I is the interval $[0,3]$ on the real axis. Since $\int_I x\,dz = \int_0^3 x\,dx = \frac{9}{2}$, we know that $\int_C x\,dz = -\int_D x\,dz + \int_I x\,dz = -i\left(\frac{5}{2}\right) + \frac{9}{2}$. This is the same answer!

(16) 5. a) Suppose that $h(x,y)$ is a harmonic function. Verify that $\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}$ is analytic.
Answer If $u = h_x$ and $v = -h_y$ then the equation $u_x = v_y$ is just $h_{xx} = -h_{yy}$ which is $h_{xx} + h_{yy} = 0$: this is true since h is harmonic. The equation $u_y = -v_x$ is $h_{xy} = -(-h_{yx})$ which is true since mixed partial derivatives are equal. The Cauchy-Riemann equations are satisfied, so $u + iv$ is analytic.
b) Verify that $3x^2y - y^3 + y$ is harmonic, and find all harmonic conjugates.
Answer Suppose $u = 3x^2y - y^3 + y$. Then $u_x = 6xy$ and $u_{xx} = 6y$, and $u_y = 3x^2 - 3y^2 + 1$ and $u_{yy} = -6y$. Therefore $u_{xx} + u_{yy} = 6y - 6y = 0$, and u is harmonic. If v is a harmonic conjugate, then $v_y = u_x = 6xy$ so that $v = 3xy^2 + f(x)$. Then $v_x = 3y^2 + f'(x)$ which should be the same as $-u_y = -(3x^2 - 3y^2 + 1) = -3x^2 + 3y^2 - 1$. Since $f'(x) = -3x^2 - 1$ we know $f(x) = -x^3 - x + K$ (K is a constant) so $v = 3xy^2 - x^3 - x + K$.
Note $u + iv$ is $-iz^3 - iz$, readily identifiable as analytic.

(12) 6. a) Define $\sin(z)$ and $\cos(z)$ in some way appropriate for this course, and use your definitions to prove that $\sin(2z) = 2\sin(z)\cos(z)$ for all complex z. **Answer** Use these definitions: $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$. Then $\sin(z)\cos(z) = \frac{(e^{iz})^2 - (e^{-iz})^2}{4i} = \frac{e^{2iz} - e^{-2iz}}{4i} = \frac{1}{2}\sin(2z)$ and we're done.
b) Find a value of z so that $|\sin z| > 2$. Verify your assertion. **Answer** If $z = 10i$, then $\sin(10i) = \frac{e^{-10} - e^{10}}{2i}$ so $|\sin(10i)| \overset{\nabla}{\geq} \frac{e^{10}}{2} - \frac{e^{-10}}{2} > \frac{2^{10}}{2} - 1 = 511$.

(12) 7. Verify that $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz} - 3z}{z^4 + 100}\,dz = 0$ where C_R is the semicircle of radius R and center 0 in the upper halfplane going from R to $-R$ as shown.



Answer If $z = x + iy$ with $y \geq 0$, then $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = |e^{ix}e^{-y}| = |e^{ix}||e^{-y}| = e^{-y} \leq 1$. Then $|e^{iz} - 3z| \overset{\triangle}{\leq} |e^{iz}| + |3z| \leq 1 + 3R$ when z is on C_R. Also, $|z^4 + 100| \overset{\nabla}{\geq} |z|^4 - 100 = R^4 - 100$. We know $R^4 > 100$ (and therefore division by 0 doesn't occur!) when, say, $R > \sqrt{10}$. Now let's use ML on the integral. The length of the curve is πR. The modulus of the integrand is overestimated by an upper bound on the top of the fraction divided by a lower bounded on the bottom of the fraction. Therefore $\left|\int_{C_R} \frac{e^{iz} - 3z}{z^4 + 100}\,dz\right| \leq (\pi R)\left(\frac{1 + 3R}{R^4 - 100}\right)$. Since the degree of the top is 1 and the degree of the bottom is 4, I think this quotient $\to 0$ as $R \to \infty$ and therefore the integral also $\to 0$ as $R \to \infty$

(12) 8. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \left(\frac{n}{4^n}\right) z^n$ and find a "closed form" (that is, a simple expression) for the sum of this power series. Use this closed form to compute the exact value of the sum $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$. **Answer** The sum converges absolutely inside its radius of convergence and diverges outside, so we need just check for absolute convergence. We use the Ratio Test. Take $a_n = \frac{n}{4^n} z^n$ so $a_{n+1} = \frac{n+1}{4^{n+1}} z^{n+1}$. Then $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)z}{4n}\right| = \left(1 + \frac{1}{n}\right)\left|\frac{z}{4}\right|$. This $\to \frac{|z|}{4}$ as $n \to \infty$. We get divergence when $|z| > 4$ and absolute convergence when $|z| < 4$. The radius of convergence is 4.
If $f(z) = \sum_{n=1}^{\infty} \left(\frac{n}{4^n}\right) z^n$, then $f(z) = z \sum_{n=1}^{\infty} \left(\frac{n}{4^n}\right) z^{n-1}$. But $\sum_{n=1}^{\infty} \left(\frac{n}{4^n}\right) z^{n-1}$ is the derivative (inside its radius of convergence) of $\sum_{n=1}^{\infty} \left(\frac{1}{4^n}\right) z^n$. This is, in turn, a geometric series with first term $\frac{z}{4}$ and ratio also $\frac{z}{4}$. Its sum is $\frac{\frac{z}{4}}{1 - \frac{z}{4}} = \frac{z}{4 - z}$. Therefore $f(z)$ is the derivative of this function multiplied by z. Now $\frac{d}{dz}\left(\frac{z}{4 - z}\right) = \frac{(4 - z) - (-z)}{(4 - z)^2} = \frac{4}{(4 - z)^2}$ and therefore $f(z) = \frac{4z}{(4 - z)^2}$. The sum $\sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n$ is $f(3)$, and this is 12.