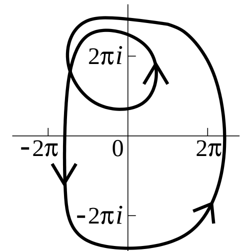
Here are answers that would earn full credit. Other methods may also be valid.

(18) 1. Suppose $f(z)=\frac{1}{z^2(e^z-1)}$.



a) Find and classify (removable, pole, essential) *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

Answer 0 is a pole of order 3 since $e^z-1=\sum_{n=1}^{\infty}\frac{z^n}{n!}=z\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}=zg(z)$ where the sum indicated, which converges for all z, defines a function whose value at 0 is $1\neq 0$. Therefore $f(z)=\frac{1}{z^3}h(z)$, where $h(z)$ is analytic near 0 (it is $\frac{1}{g(z)}$) and is not 0 at 0. This verifies the assertion about the order of the pole. So $f(z)=\frac{A}{z^3}+\frac{B}{z^2}+\frac{C}{z}+\ldots$ and we want C. Multiply by z^3 to get $\frac{z}{e^z-1}=A+Bz+Cz^2+\ldots$. Then some division (divide 1 by $1+\frac{1}{2}z+\frac{1}{6}z^2+\ldots$ and look at the third term) gives $\frac{1}{12}$ as the value of the residue

$e^z=1$ also for all $2\pi n$ where n is a non-zero integer. Look at $z^2(e^z-1)$: its derivative is $2z(e^z-1)+z^2(e^z)$ which at $2\pi ni$ is $-4\pi^2n^2$ which is not 0 if $n\neq 0$. Therefore $z^2(e^z-1)$ has a first order zero for such z's. The residue is $\lim_{z\to 2\pi n}=\frac{z-2\pi n}{z^2(e^z-1)}=-\frac{1}{4\pi^2n^2}$ by, say, L'Hôpital's rule.

b) Compute $\int_C f(z)\,dz$ where C is the closed curve displayed to the right.

Answer We can break this curve into two simple closed curves. Let C_1 be the small loop just going around $2\pi i$. The Residue Theorem applies, and the only singularity inside that curve is at $2\pi i$. The integral over C_1 is $(2\pi i)\left(-\frac{1}{4\pi^2}\right)=-\frac{i}{2\pi}$. Now consider C_2, what's left. This curve has the singularities at $-2\pi i$, 0, and $2\pi i$ inside it. The integral over C_2 can be computed using the Residue Theorem: $2\pi i\left(-\frac{1}{4\pi^2}+\frac{1}{12}-\frac{1}{4\pi^2}\right)=-\frac{i}{\pi}+\frac{\pi i}{6}$. The integral over C is the sum: $-\frac{3i}{2\pi}+\frac{i\pi}{6}$.

(18) 2. Prove that for any fixed complex number w, $\frac{1}{2\pi}\int_0^{2\pi}e^{2w\cos\theta}\,d\theta=\sum_{n=0}^{\infty}\left(\frac{w^n}{n!}\right)^2$.

Answer If $q=e^{i\theta}$ then $\cos\theta=\frac{q+q^{-1}}{2}$. The series for the exponential function can be used, and we get $e^{2w\cos\theta}=\sum_{n=0}^{\infty}\frac{1}{n!}(2w)^2\left(\frac{q+q^{-1}}{2}\right)^n$. Of course, $dq=e^{i\theta}i\,d\theta$ so $\frac{dq}{iq}=d\theta$. Stuff everything together and interchange sum and integral, and we need to compute: $\sum_{n=0}^{\infty}\frac{1}{n!}(w^n)\frac{1}{i}\int_{|z|=1}\left(q+q^{-1}\right)^n\frac{dq}{q}$.

Now try some experiments with $\int_{|z|=1}\left(q+q^{-1}\right)^n\frac{dq}{q}$. The only powers of q which "matter" (give a non-zero integral) are the negative first powers. This means we need q^0 from $\left(q+q^{-1}\right)^n$. So $\left(q+q^{-1}\right)^3=q^3+3q+3q^{-1}+q^{-3}$ and $\left(q+q^{-1}\right)^4=q^4+4q^2+6+4q^{-2}+q^{-4}$. We get q^0 in $\left(q+q^{-1}\right)^n$ when n is even, say $n=2k$. The coefficient we get is the central binomial coefficient, $\frac{(2k)!}{k!k!}$. Back to the series, remembering that integration multiplies by $2\pi i$:

$\sum_{n=0}^{\infty}\frac{1}{n!}(w^n)\frac{1}{i}\int_{|z|=1}\left(q+q^{-1}\right)^n\frac{dq}{q}=$ (Only *even* n's!) $\sum_{n=0}^{\infty}\frac{1}{n!}(w^n)\frac{1}{i}2\pi i\frac{(2k)!}{k!k!}$. Switch from n's to k's, cancel the i and notice that $n!$ is the same as $(2k)!$ and get $2\pi\sum_{k=0}^{\infty}\frac{w^{2k}}{(k!)^2}$. This gives the formula desired.

Student solutions Some students wrote *much better answers*. They remarked that the integrand is $e^{wq}e^{w/q}$ which has an isolated singularity at $q=0$, and we only need to compute the residue at 0 of this function. But this means "picking out" the -1^{st} coefficient of q in the product $\frac{1}{q}\left(\sum_{n=0}^{\infty}\frac{w^nq^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{w^n}{q^nn!}\right)$ and this leads immediately to the formula requested: really neater than what I wrote above.

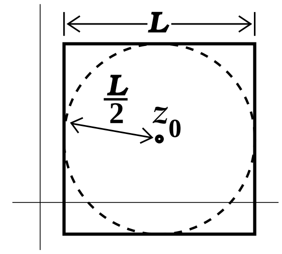
(14) 3. Show that if $f(z)$ is analytic near a and $g(z)=\frac{f(z)+af'(a)-zf'(a)-f(a)}{(z-a)^2}$ then $g(z)$ has a removable singularity at $z=a$. What value should be given to $g(a)$ so that the extended function is analytic at a?

Answer Here **h.o.t.** will mean higher order terms in $z-a$ added together in a series which converges near a. Near a, $f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\textbf{h.o.t.}$. We will plug this into the formula for $g(z)$ when z is near a but not equal to a (otherwise we would divide by 0):

$g(z)=\frac{f(a)+f'(a)(z-a)+\frac{f''(a)}{2}(z-a)^2+\textbf{h.o.t.}+af'(a)-zf'(a)-f(a)}{(z-a)^2}=\frac{\frac{f''(a)}{2}(z-a)^2+\textbf{h.o.t.}}{(z-a)^2}=\frac{f''(a)}{2}+\textbf{h.o.t.}$ and we see that indeed $g(z)$ is given by a convergent power series near $z=a$, and the value of the series at $z=a$ is $\frac{f''(a)}{2}$, so that should be $g(a)$.

Student solutions If we can just show that $\lim_{z\to a} g(z)$ exists and is finite, then the Riemann Removable Singularity Theorem will apply. Also, the limit will be the desired value of $g(a)$. Two applications of L'Hôpital's Rule do this very efficiently but L'H eligibility (twice of the form $0/0$) should be mentioned.

(12) 4. Suppose $|f(z)| \le K$ on the circumference of a square whose side length is L, and let z_0 be the center of the square. If $f(z)$ is analytic in a domain containing the square, show that $|f'(z_0)| \le \frac{8K}{\pi L}$.



Answer Let S be that square. The Cauchy Integral Formula for the first derivative declares that $f(z_0) = \frac{1}{2\pi i}\int_S \frac{f(z)}{(z-z_0)^2}dz$. Now apply the ML inequality. The length of S is $4L$. The modulus of $f(z)$ on S is bounded by K. If z_0 is the center, and z is on the square, then $|z - z_0| \ge \frac{L}{2}$. Combining all of these estimates, we see that $|f(z_0)| \le (4L)\frac{1}{2\pi}K\left(\frac{2}{L}\right)^2 = \frac{8K}{\pi L}$ as desired.

(12) 5. Prove: there is exactly one entire function $f(z)$ which satisfies $|f(z)| \le \sqrt{|z|}$ for all z.

Answer The entire function $Z(z) = 0$ for all z satisfies this estimate. Now suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function with $|f(z)| \le \sqrt{|z|}$ for all z. Because $f(0) = a_0$ and $\sqrt{0} = 0$ we know $a_0 = 0$. Since $a_n = \frac{f^{(n)}(0)}{n!}$, and we know the Cauchy estimates: $|f^{(n)}(0)| \le \frac{n!}{r^n}\max_{|z|=r}|f(z)|$, we combine the information to deduce $|a_n| \le \frac{1}{r^n}\sqrt{r}$. If n is a positive integer, the inequality $|a_n| \le r^{\frac{1}{2}-n}$, valid for all positive r, shows that $a_n = 0$ since $\frac{1}{2} - n < 0$ so $\lim_{r\to\infty} r^{\frac{1}{2}-n} = 0$. Therefore all of the a_n's are 0, and $f(z) = Z(z)$.

(16) 6. a) Find the Laurent series for $\frac{1}{z^2}$ in the annulus $1 < |z - i| < \infty$.

Answer $\frac{1}{z} = \frac{1}{i+(z-i)} = \left(\frac{1}{z-i}\right)\left(\frac{1}{\frac{i}{z-i}+1}\right) = \left(\frac{1}{z-i}\right)\sum_{n=0}^{\infty}\left(-\frac{i}{z-i}\right)^n = \sum_{n=0}^{\infty}\frac{(-i)^n}{(z-i)^{n+1}}$. The expansion is valid (geometric series) since in this region, $\left|-\frac{i}{z-i}\right| = \frac{1}{|z-i|} < 1$. Differentiate and multiply by -1, to get $\frac{1}{z^2} = \sum_{n=0}^{\infty}\frac{(-i)^n(n+1)}{(z-i)^{n+2}}$. The series begins $\frac{1}{(z-i)^2} - \frac{2i}{(z-i)^2} - \frac{3}{(z-i)^3} - \frac{4i}{(z-i)^4} - \frac{5}{(z-i)^5} + \ldots$

b) Find the Taylor series centered at i for $\mathrm{Log}(z)$. What is its radius of convergence?

Answer The derivative of $\mathrm{Log}(z)$ is $\frac{1}{z}$ so that we just need a Taylor series for $\frac{1}{z}$ centered at i, and then adjust the antiderivative by a constant. Note that $\frac{1}{z}$ has a pole at 0, so all the series considered here will have 1 as radius of convergence: the distance from the center to the nearest singularity. So $\frac{1}{z} = \frac{1}{i+(z-i)} = \frac{1}{i}\left(\frac{1}{1-i(z-i)}\right) = \frac{1}{i}\sum_{n=0}^{\infty} i^n(z-i)^n = \sum_{n=0}^{\infty} i^{n-1}(z-i)^n$ and we antidifferentiate (using the value $\mathrm{Log}(i) = i\frac{\pi}{2}$) to get $\mathrm{Log}(z) = i\frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{i^{n-1}}{n+1}(z-i)^{n+1}$, valid for $|z - i| < 1$.

(10) 7. Suppose $f(z)$ is analytic in all of $\mathbb{C}$ except the closed negative real axis (that's where x is real and $x \le 0$). If $f(x) = x^x$ for real positive x, find $f(i)$.

Answer Notice that $x^x = e^{x\ln(x)}$ for $x > 0$, so the analytic function defined by $g(z) = e^{z\mathrm{Log}(z)}$ (analytic since it is a composition of analytic functions), whose domain is all of $\mathbb{C}$ except the closed negative real axis, is equal to $f(z)$ on the positive reals. Since this is a set with an accumulation point, $f(z) = e^{z\mathrm{Log}(z)}$ for all z's where $f(z)$ is defined. So we can compute $f(i)$ by using the formula given in the definition for $g(z)$: $f(i) = e^{i\mathrm{Log}(i)} = e^{i\left(\frac{i\pi}{2}\right)} = e^{-\frac{\pi}{2}}$.

Sources

1. Problem #4 on my final exam in Math 503 (graduate complex variables at Rutgers) in fall 2007.
2. University of California at Berkeley qualifying exam problem.
3. Temple University qualifying exam problem.
4. University of Missouri qualifying exam problem.
5. Me (resembling part of a more difficult problem #5 on my final exam in Math 503 in fall 2004).
6. a) is from a book by Bruce Palka which I used once as a text in 503, and b) is from me.
7. Most of a Temple University qualifying exam problem.