Second exam

Due at the beginning of class, 5 PM, on Monday, April 21.

Rules

1. You may not discuss or communicate with any other person about this exam until after the end of class, 6:20 PM, on Monday, April 21.

2. You may use any other sources for help. For example, textbooks, notes, "the Internet", and computers or calculators. However, any answer must be fully supported by your own reasoning. For example, the values of many definite integrals can easily be obtained with computer help. The value itself will receive minimal credit, and full credit can be earned only by appropriate citations of theorems and use of particular estimates and limits.

3. If you have any questions about these rules, please let me know. I think the problems are understandable so no "hints" will be given (it is an exam!).

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(18) 1. Suppose $f(z) = \dfrac{1}{z^2 (e^z - 1)}$.

a) Find and classify (removable, pole, essential) *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

b) Compute $\int_C f(z)\, dz$ where C is the closed curve displayed to the right.
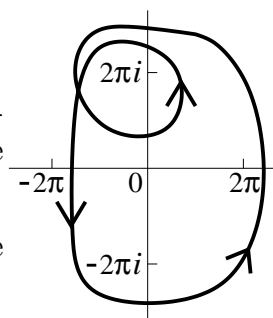
(18) 2. Prove that for any fixed complex number w, $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} e^{2w \cos \theta}\, d\theta = \sum_{n=0}^{\infty} \left(\dfrac{w^n}{n!} \right)^2$.

(14) 3. Show that if $f(z)$ is analytic near a and $g(z) = \dfrac{f(z) + af'(a) - zf'(a) - f(a)}{(z-a)^2}$ then $g(z)$ has a removable singularity at $z = a$. What value should be given to $g(a)$ so that the extended function is analytic at a?

(12) 4. Suppose $|f(z)| \le K$ on the circumference of a square whose side length is L, and let z_0 be the center of the square. If $f(z)$ is analytic in a domain containing the square, show that $|f'(z_0)| \le \dfrac{8K}{\pi L}$.

Hint Use an integral formula.

(12) 5. Prove: there is exactly one entire function $f(z)$ which satisfies $|f(z)| \le \sqrt{|z|}$ for all z.

(16) 6. a) Find the Laurent series for $\dfrac{1}{z^2}$ in the annulus $1 < |z - i| < \infty$.

b) Find the Taylor series centered at i for $\mathrm{Log}(z)$. What is its radius of convergence?

(10) 7. Suppose $f(z)$ is analytic in all of $\mathbb{C}$ except the closed negative real axis (that's where x is real and $x \le 0$). If $f(x) = x^x$ for real positive x, find $f(i)$.