(20) 1. a) Explain why there is exactly one entire function $f(z)$ with the property that $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers, n. What is this $f(z)$?

b) Exhibit explicitly two distinct entire functions $g(z)$ and $h(z)$ with the property that the value of these functions at n is n^2 for all positive integers, n.

(20) 2. Suppose C is a circle with center $3+i$ and radius 3. Compute the following integrals with brief reasoning supporting each computation.

a) $\displaystyle\int_C \frac{\cos z}{z^2-\pi^2}\,dz$ b) $\displaystyle\int_C \frac{e^{5z}}{(z-4)^{15}}\,dz$ c) $\displaystyle\int_C \frac{\sin z}{z^2+\pi^2}\,dz$

(20) 3. Use complex analysis to show $\displaystyle\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{\pi}{\sqrt{3}}$. Show clearly any contour of integration and any residue computation. Explain why the limiting value of some integral is 0.

(20) 4. a) If k is a real number with $-1 < k < 1$, derive the Laurent series representation $\displaystyle\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n}$ (for z's so that $|k| < |z| < \infty$).

b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula $\displaystyle\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k\cos\theta - k^2}{1-2k\cos\theta+k^2}$.

(20) 5. If $f(z) = \dfrac{\sin(z^2)}{(\sin z)^2}$, find and classify *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

(20) 6. Find $r > 0$ so that $P(z) = z^3 - 4z^2 + z - 4$ has exactly 2 roots inside the circle $|z| = r$.

(20) 7. a) Suppose that $h(x,y)$ is a harmonic function. Verify that $\dfrac{\partial h}{\partial x} - i\dfrac{\partial h}{\partial y}$ is analytic.

b) There is one value of A so that $Ax^2y - y^3 + \sin(4x)\cosh(4y)$ is harmonic. Find that value of A, and find all harmonic conjugates of that harmonic function.

(20) 8. Suppose $f(z)$ is a function which is analytic in all of $\mathbb{C}$ except for poles at $2i$ and $4i$ and $-3i$. Further, suppose that $f(z)$ is real when z is real. The Taylor series centered at $x = 5$ for this function is considered in a calculus class: this is the series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n$. There will be a largest positive real number R so that this series converges absolutely for all real numbers x satisfying $5 - R < x < 5 + R$. What is this number R? Give brief reasoning supporting your assertion: you should explain why there is such a number, and explain why the R given is the largest such number.

(20) 9. Exactly one linear fractional transformation, $T(z)$, maps i to -1, 0 to 0, and 1 to 1.
Note In this problem, you need only show computations. No further explanation is needed.

a) Find a formula for $T(z)$.

b) What is $T(\infty)$? **Answer** $T(\infty) =$ ______

c) What is the image of $\mathbb{R} \cup \{\infty\}$ (the x-axis and infinity, the *real axis*) under this linear fractional transformation? Sketch this image set (which is in the range) on the axes to the right and label it with the letter **A**. Write a brief description of this set in the space below.

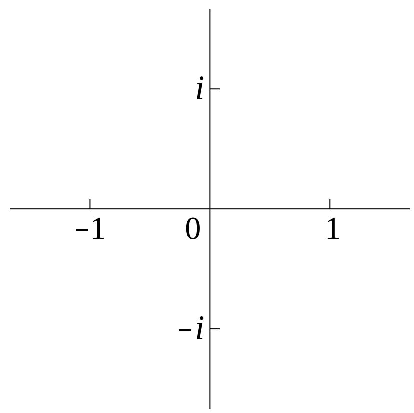


Image plane, w, if $w = T(z)$

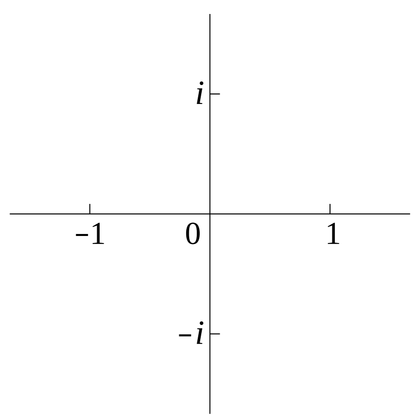
____

d) What is the image of $i\mathbb{R} \cup \{\infty\}$ (the y-axis and infinity, the *imaginary axis*) under this linear fractional transformation? Sketch this image set (which is in the range) on the axes to the right and label it with the letter **B**. Write a brief description of this set in the space below.

____

e) Find z so that $T(z) = \infty$. **Answer** $z =$ ______

f) What is the set of z's whose image under the mapping $T(z)$ is $\mathbb{R} \cup \{\infty\}$ (the x-axis and infinity)? Sketch this set (which is in the domain) on the axes provided to the right. Write a brief description of this set in the space below.



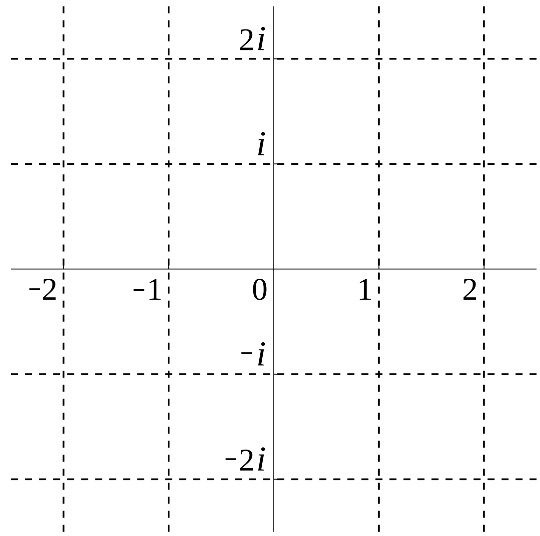
Domain plane, z

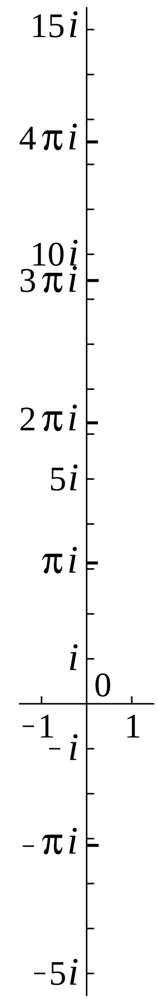
____

(20) 10. The two parts of this problem are not related.

Note In this problem, you need only show computations. No further explanation is needed.

a) Describe all solutions of $z^4 = -16$ algebraically in rectangular form. Sketch the solutions on the axes provided.





b) Describe all values of $\log(-\sqrt{e})$ algebraically in rectangular form. Sketch as many solutions as you can on the axes provided.

Final Exam for Math 403, section 1

May 9, 2008

NAME ____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
No notes, texts, or calculators may be used on this exam.

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total Points Earned:		