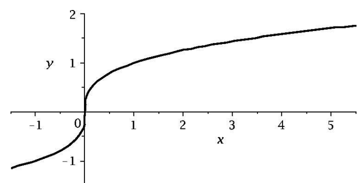
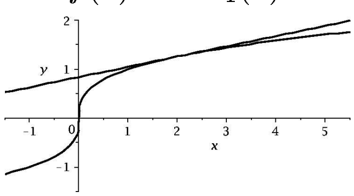
Math 403, section 1 **Power series and Fourier series** **March 4, 2008**

Power series are magically nice. Such a series can be differentiated "termwise" inside its radius of convergence, and there the sum of the differentiated series turns out to be the derivative of the sum of the original power series. That's true because the geometric series converges so very, very nicely: $\sum_{n=0}^{\infty} r^n$ is terrific when $|r| < 1$, and then $\sum_{n=0}^{\infty} nr^{n-1}$ also converges (Ratio Test!), and, in fact, if $P(n)$ is *any* polynomial of any degree, $\sum_{n=0}^{\infty} P(n)r^n$ also converges if $|r| < 1$. Partial sums of power series are *Taylor polynomials* of the function: $P_N(x) = \sum_{n=0}^{N} \frac{f^n(0)}{n!}(x - x_0)^n$. These polynomials can be qualitatively characterized as the *polynomial approximation of the given degree which is closest near the specified point.*

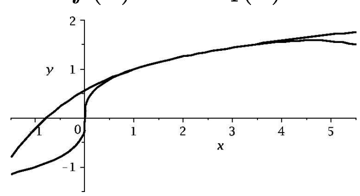
For example, take $f(x) = x^{1/3}$, the cube root of x. Let's look at the function and some of its Taylor polynomials centered at $x_0 = 2$. A graph of f alone in the window $-1.5 \le x \le 5.5$ and $-1.5 \le y \le 2$ is shown to the right,



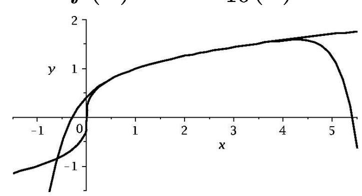
$f(x)$ and $P_1(x)$



$f(x)$ and $P_4(x)$

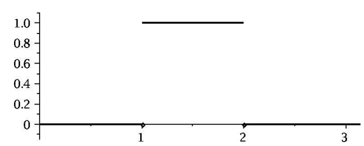


$f(x)$ and $P_{10}(x)$

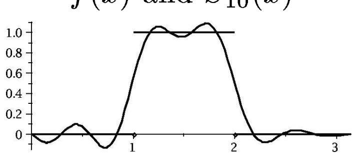


Fourier series (I'll just look at what are called *Fourier sine series* here) have a different characterization. These series have partial sums which are the *best mean square approximations on an interval.* If f is defined on $[0, \pi]$, its Fourier sine coefficients are given by $a_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(mx)\,dx$ (m is a positive integer). The Fourier series of f is $\sum_{m=1}^{\infty} a_m \sin(mx)$. A partial sum $S_M(x) = \sum_{m=1}^{M} a_m \sin(mx)$ has this property: the numbers $a_1, a_2, \dots, a_M$ are those for which $\int_0^{\pi} \big(f(x) - S_M(x)\big)^2 dx$ is smallest.

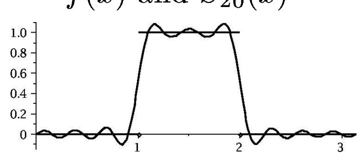
Suppose that f is the function which is 1 in the interval $[1, 2]$ and 0 otherwise. This function has two values, but it is not continuous. A graph of the function in the window $0 \le x \le \pi$ and $-.15 \le y \le 1.1$ is shown to the right. The Fourier series of this function converges for all x.



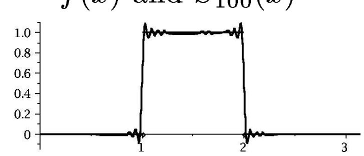
$f(x)$ and $S_{10}(x)$



$f(x)$ and $S_{20}(x)$

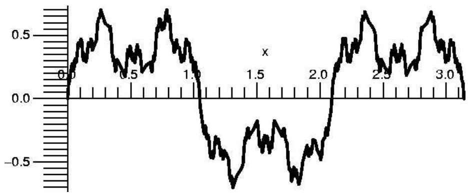


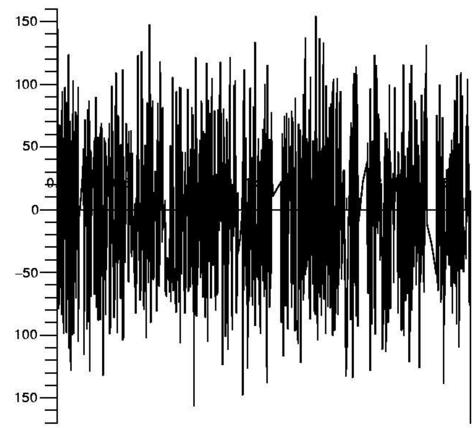
$f(x)$ and $S_{100}(x)$



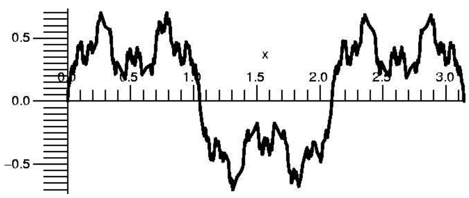
The sum of the whole infinite Fourier series is *not* continuous. In fact, it is equal to f except at 1 and 2, where the value of the sum is $\frac{1}{2}$. The Fourier series averages the right- and left-hand limits at such jumps. These statements are not obvious!

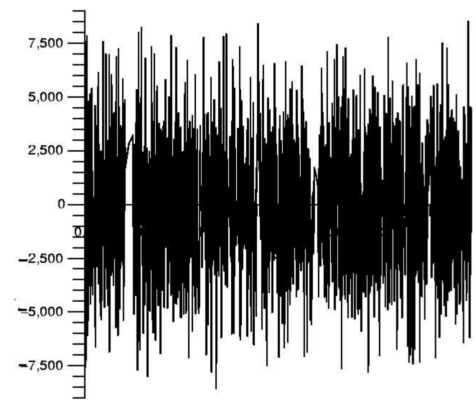
Consider this Fourier sine series: $f(x) = \sum_{j=1}^{\infty} \frac{1}{2^j} \sin(3^j x)$. The series of amplitudes, outside of the sines, certainly converges. The sine function (real x here!) always has values between -1 and 1, so the whole series converges absolutely and therefore converges. Reasoning similar to what we did for power series will show that this f is continuous. Below are some pictures of partial sums and their derivatives. Pay close attention to the vertical scales of the graphs, because they are nearly *unbelievable!*



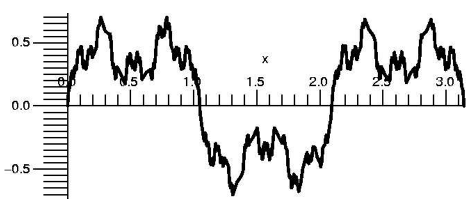


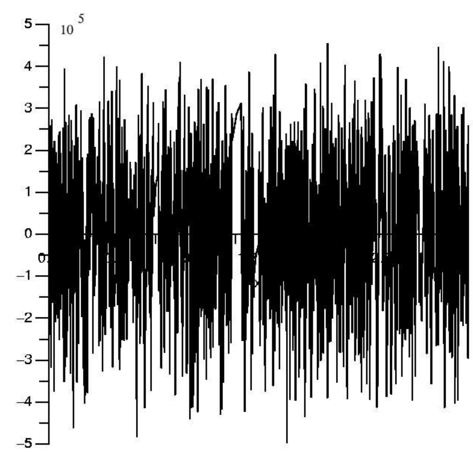
Above: the graph of S_{10}; to the right: the graph of $\frac{dS_{10}}{dx}$ which is $\sum_{j=1}^{10} \frac{3^j}{2^j} \cos(3^j x)$.





Above: the graph of S_{20}; to the right: the graph of $\frac{dS_{20}}{dx}$ which is $\sum_{j=1}^{20} \frac{3^j}{2^j} \cos(3^j x)$.





Above: the graph of S_{30}; to the right: the graph of $\frac{dS_{30}}{dx}$ which is $\sum_{j=1}^{30} \frac{3^j}{2^j} \cos(3^j x)$.

f is an example of a function which is continuous at every point and differentiable at no point. The names associated with such functions: Robert Brown (1827) observed the motion of pollen and "saw" such graphs; Bachelier (1900) connected Brownian motion with variations in stock and option markets; one of Einstein's famous results of 1905 explained Brownian motion using probability (particles of dust move as a result of random molecular collisions, which is heat). Most academic mathematicians of the late nineteenth century were quite reluctant to admit that such functions could exist!