Math 403:01 **Possible final exam problems** April 28, 2008

Your final exam will have 10 problems. At least 5 will be chosen from this list.

1. a) Explain why there is exactly one entire function $f(z)$ with the property that $f\left(\frac{1}{n}\right) = \frac{1}{n^2}$ for all positive integers, n. What is this $f(z)$?

b) Exhibit explicitly two distinct entire functions $g(z)$ and $h(z)$ with the property that the value of these functions at n is n^2 for all positive integers, n.

2. Suppose that an entire function is real-valued. Explain why it must be constant.

3. Suppose C is a circle with center $3+i$ and radius 3. Compute $\int_C \frac{\cos z}{z^2-\pi^2}\,dz$, $\int_C \frac{e^{5z}}{(z-4)^{15}}\,dz$, and $\int_C \frac{\sin z}{z^2+\pi^2}\,dz$ with brief reasoning supporting each computation.

4. Use complex analysis to show $\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{\pi}{\sqrt{3}}$. Show clearly any contour of integration and any residue computation. Explain why the limiting value of some integral is 0.

5. a) If k is a real number with $-1 < k < 1$, derive the Laurent series representation $\frac{k}{z-k} = \sum_{n=1}^{\infty} \frac{k^n}{z^n}$ (for z's so that $|k| < |z| < \infty$).

b) Write $z = e^{i\theta}$ in the equation obtained in part a) and then equate real parts on each side of the result to derive the summation formula $\sum_{n=1}^{\infty} k^n \cos n\theta = \frac{k\cos\theta - k^2}{1-2k\cos\theta+k^2}$.

6. What is the radius of convergence of the Taylor series of $f(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}$ centered at the point $z = i$? Briefly explain your answer.

7. If $f(z) = \frac{\sin(z^2)}{(\sin z)^2}$, find and classify *all* isolated singularities of f. If the isolated singularity is a pole, tell the order of the pole and the residue of f at the pole.

8. Find $r > 0$ so that $P(z) = z^3 - 4z^2 + z - 4$ has exactly 2 roots inside the circle $|z| = r$.

9. Student Wilbur wants to define the arctangent function directly using complex analysis. He believes that $\arctan(0)$ should be 0 and that the derivative of $\arctan(z)$ should be $\frac{1}{1+z^2}$. His candidate function will be called $\mathcal{W}(z)$. If z is not $\pm i$, here is his method: suppose that C is a curve from 0 to z not passing through $\pm i$. Then $\mathcal{W}(z)$ should be $\int_C \frac{1}{1+w^2}\,dw$.

a) If z is not equal to $\pm i$, explain briefly why there must be such a curve.

b) Will the result of the integration always be the same, no matter which eligible curve C is chosen? Explain why or why not.

c) If your answer to b) is that the result will not always be the same, explain how Student Wilbur could restrict the domain so that his method will be successful: give a <u>largest connected open set U containing 0</u> in which his method defines a (single-valued!) function whose derivative is $\frac{1}{1+z^2}$ and whose value at 0 is 0.

10. Let S be the open strip of complex numbers, z, defined by $\pi < \operatorname{Im} z < 3\pi$. Sketch S and then sketch and describe the image of S under each of these functions: e^z, $\frac{1}{z}$, and z^2.