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An interpretation of Biblical Chapter 2, verse 43

Theorem: let P be a nonempty, perfect set in R^k. Then P is uncountable.

Proof: P*roof by contradiction and induction.*

Let P be a nonempty, perfect set in R^k. *This means that P is closed (contains all of its limit points) and that every point in P is a limit point.*

Since P has limit points, P is infinite. *This is Theorem 2.20*

Suppose to the contrary that P is countable. Then we can denote all of the points in P by x_1, x_2, $x_{3,\ldots\ldots}$

We shall construct a set of neighborhoods $\{V_n\}$ - a sequence of closed bounded neighborhoods in R^k (so a sequence of nested compact sets) which have non empty intersections with P. Then we will reach a contradiction.

n=1: Let V_1 be any neighborhood of x_1. V_1 consists of all $y \in R^k$ s.t. $|y-x_1| < r$ for some r. Then the closure of V_1 consists of all $y \in R^k$ s.t. $|y-x_1| \leq r$. Thus $V_1 \cap P$ is nonempty- this is the induction hypothesis.

arbitrary n: Suppose V_n is a neighborhood in R^k that has been constructed so that $V_n \cap P$ is not empty- so there exists some x_n of P in V_n. *There are open neighborhoods around x_n which are subsets of V_n.*

inductive step, n+1: Now we can see that we can pick a neighborhood V_{n+1} such that

(i) closure of(V_{n+1})$\subset V_n$. *It is important to note that both V and V_{n+1} are compact (neighborhoods with finite radii and including their "shells")*

(ii) $x_n \notin V_{n+1}$

(iii) $V_{n+1} \cap P$ is not empty. (iii) is exactly what satisfies our induction hypothesis.

Now, to find our contradiction we construct sets K_n= closure of $(V_n) \cap P$. P is perfect, so it is closed, and the closure of V_n is closed and bounded so it is compact. From the Corollary to Theorem 2.35 on pg38, we know that the intersection of a closed set and a compact set is compact. Thus K_n is compact.

We know that K_n is not empty because $V_n \cap P$ closure of $(V_n) \cap P$ contains the nonempty *from (iii)* set $V_n \cap P$.

Because of our particular construction $x_n \notin V_{n+1}$- thus $x_n \notin K_{n+1}$, and since P is countable, every point in P is NOT in one of neighborhoods, thus no point of P lies in $\bigcap_1^{\infty} Kn$

Since $K_n \subset P$, this implies that $\bigcap_1^{\infty} Kn$ is empty. *Remember that K_n was defined as the intersection between P and a neighborhood of the reals.*

But, we also proved that each K_n is nonempty and compact.

And we know that $K_{n+1} \subset K_n$

Pf: if x_{n+1} is in K_{n+1}, then x_{n+1} is in V_{n+1} and x_{n+1} is in P. Since x_{n+1} is in V_{n+1}, and V_{n+1} is a subset of V_n, we know that x_{n+1} is in $V_{n.}$ Thus x_{n+1} is in both V_n and in P; thus x_{n+1} is in K_n. This means that K_{n+1} is a subset of K_n.

And we know that Corollary to Theorem 2.36 says that: If $\{K_n\}$ is a sequence of nonempty, compact sets such that $K_{n+1} \subset K_n$, then $\bigcap_1^{\infty} Kn$ is nonempty. *This means that the intersection of an infinite sequence of nested compact sets is nonempty.*

We now that have $\bigcap_1^{\infty} Kn$ is empty and that $\bigcap_1^{\infty} Kn$ cannot be empty. Contradiction!

Thus P must be an uncountable set.