Hand in solutions to these problems on Monday, December 1. You may use any sources but please do not discuss your solutions with other people. Solutions should be brief and clear.

(20) 1. True or false. If true, give a very brief explanation of why the statement is correct. If false, supply an example showing why the implication is false.

a) If $\{a_n\}$ is a complex sequence for which $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

TRUE OR FALSE? ________

b) A closed subset of a complete metric space is complete.

TRUE OR FALSE? ________

c) A continuous real-valued function which is bounded below on a complete metric space achieves its minimum.

TRUE OR FALSE? ________

d) A closed subset of a connected metric space is connected.

TRUE OR FALSE? ________

e) If $\{a_n\}$ is a complex sequence for which $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

TRUE OR FALSE? ________

(16) 2. If $f\colon (0,1] \to \mathbb{R}$ is continuous, prove that f is *uniformly continuous* if and only if $\lim_{x\to 0} f(x)$ exists.

(18) 3. a) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has two continuous derivatives. That is, both f' and f'' exist everywhere and are continuous. Suppose $a \in \mathbb{R}$. Prove: if $f'(a) = 0$ and $f''(a) < 0$ then f has a local maximum at a. Actually, such an f has a *strict* local maximum at a.

b) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ has 68 continuous derivatives. That is, if $1 \le j \le 68$, then $f^{(j)}$ exists everywhere and is continuous. Suppose $a \in \mathbb{R}$. Prove: if $f^{(j)}(a) = 0$ for all j with $1 \le j \le 67$ and $f^{(68)}(a) < 0$ then f has a strict local maximum at a.

c) Suppose the previous problem had the hypotheses $f\colon \mathbb{R} \to \mathbb{R}$ has 67 continuous derivatives, $f^{(j)}(a) = 0$ for all j with $1 \le j \le 66$ and $f^{(67)}(a) < 0$. What would be the local behavior of f near a?

(16) 4. Suppose that f is a continuous real-valued function on $[0,1]$. Define g on $[0,1]$ by $g(x) = \sup\{\, f(t) \mid t \in [0,x] \,\}$. Prove that g is a continuous real-valued function on $[0,1]$.

(15) 5. Prove that a compact metric space is complete.

(15) 6. Suppose X and Y are metric spaces, and $f\colon X \to Y$. f is *proper* if $f^{-1}(K)$ is a compact subset of X whenever K is a compact subset of Y.

Prove: if $f\colon R^n \to [0,+\infty)$ is proper and continuous, then f attains its minimum value.