The Cartesian product of finitely many compact spaces is compact.

November 9, 2008

We will prove that the product of two compact spaces is compact; the theorem follows by induction for any finite product.

Suppose that we are given compact metric spaces (X, d_X) and (Y, d_Y). Please note that the product, $X \times Y$, will be a metric space under the metric D, such that $D((x, y), (w, z)) = \max(d_X(x, w), d_Y(y, z))$. You can check that this is a metric (though hardly anyone did on Homework 2). It also can be shown that, for $\mathbb{R}^2$, this metric is equivalent to the standard, Euclidean metric (again, this could have been shown in Homework 2). Note that in this metric, open neighborhoods will be boxes, that is, products of X-neighborhoods with Y-neighborhoods.

And, now, for the proof. Let $\mathcal{C}$ be an open covering of $X \times Y$ by open neighborhoods under D. First, note that, for any $y \in Y$, $X \times \{y\}$ is compact since it is the image of a continuous map f_y such that $f_y(x) = (x, y)$. So, we only need a finite amount of balls in $\mathcal{C}$ to cover anything of the form $X \times \{y\}$. Call such a subcover for $X \times \{y\}$, $S^y = \bigcup_{m=1}^{k_y} U_m^y \times V_m^y$ (where each U_m^y is a neighborhood in X and each V_m^y is a neighborhood in Y). Now, cover $X \times \{y\}$, for every y, by such a S^y. The collection of all such S^y's is a cover of $X \times Y$, now we need to restrict the number of y's we used to make the cover. For each S^y, take $V^y = \bigcap_{i=1}^{k_y} V_i^y$. Since this is a intersection of finitely many open sets of Y, so each V^y is an open set of Y. The collection $\{V^y | y \in Y\}$ is an open covering of Y. Since Y is compact, so there is a finite subcover $V^{y_1}, \ldots, V^{y_n}$.

Thus, $\bigcup_{j=1}^{n} S^{y_j}$ is a finite subcover of $X \times Y$. For take $(a, b) \in X \times Y$. b is in one of the V^{y_i} (since they cover Y). By the construction of V^{y_i}, S^{y_i} will contain the line $X \times \{b\}$. This is a consequence of $b \in V^{y_i}$ implying that b is in each of the $V_1^{y_i}, \ldots, V_{k_{y_i}}^{y_i}$. This, in turn, implies that the segment $(X \times \{b\}) \cap (U_j^{y_i} \times V_j^{y_i})$ is contained in each $U_j^{y_i} \times V_j^{y_i}$ for each $j \in \{1, \ldots, k_{y_i}\}$. And of course the unions of these intersections will be the line $X \times \{b\}$, which contains the point (a, b). $\square$