Math 411 Some answers to the Entrance "exam" 9/6/2008

1. (8 points) Suppose n is a positive integer. Prove that the product of any n consecutive positive integers is divisible by $n!$.

Answer Here is one possible answer. If n and m are positive integers, let

$$Q(n,m) = \frac{m(m+1)(m+2)\cdots(m+n-1)}{n!}.$$

We need to prove that $Q(n,m)$ is always an integer.

> *Easy observations* If $n = 1$, then since $1! = 1$, $Q(1,m)$ is an integer. Also, if $m = 1$, the denominator of $Q(n,m)$ is $n!$, so the quotient is 1. Therefore $Q(n,1)$ is an integer.

We have the beginning of an induction proof here. We need to decide which variable to "induct" on. I'll try to prove the following proposition:

> **A** Suppose $Q(n,m)$ is an integer for all positive integer m. Then $Q(n+1,m)$ is an integer.

I will prove that $Q(n+1,m)$ is an integer using mathematical induction. Indeed, I know that $Q(n+1,1)$ is an integer (one of the easy observations above). So I just need to prove:

> **B** Suppose $Q(n+1,m)$ is an integer for some integer m. Then $Q(n+1,m+1)$ is an integer.

Let's consider the difference between $Q(n+1,m+1)$ and $Q(n,m+1)$:

$$\frac{(m+1)(m+2)\cdots(m+n+1)}{(n+1)!} - \frac{(m+1)(m+2)\cdots(m+n)}{n!} =$$
$$\frac{(m+1)\cdots(m+n+1)(m+n+1-(n+1))}{(n+1)!} = \frac{(m+1)(m+2)\cdots(m+n+1)m}{(n+1)!}$$

This is $Q(n+1,m)$ (the m is on top at the "other" end!). Now we are assuming (**B**'s hypothesis) that $Q(n+1,m)$ is an integer, and also are assuming (**A**'s hypothesis) that $Q(n,m+1)$ is an integer. Therefore $Q(n+1,m+1)$, a sum of these two, must be an integer. So the implication **B** is correct, and therefore **A** is correct, and we have proved what's required.

Comments This proof is correct (I think!) but may be confusing. For example, I looked at both $Q(n+1,m+1) - Q(n,m+1)$ and $Q(n+1,m+1) - Q(n+1,m)$, when I was analyzing this problem. Probably a more correct approach is to realize that the numbers *count* something. They do, but that realization needs to be justified. Or you could relate $Q(n,m)$ to binomial coefficients and then declare those are integers – that also must be justified. A math cultural comment is that there are natural "objects" in mathematics which are "bigraded" (depending on, say, two integers) and one possible way of verifying properties of these objects is with a double inducation proof*.

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* For example, some of the original proofs in Several Complex Variables about the vanishing of certain cohomology groups are double induction algebraic masterpieces, considerably more complicated than what is here. Now the usual methods to verify this vanishing involve superficially simpler arguments with partial differential equations.

2. Suppose A is a non-empty subset of the positive integers, L is a real number, and $\{a_n\}$ is a sequence (a sequence is a real-valued function whose domain is the positive integers, $\mathbb{N}$). Then $\lim_{n \in A} a_n = L$ means: for all $\varepsilon > 0$ there is N in A so that if n is in A and $n > N$, then $|a_n - L| < \varepsilon$.

a) (2 points) If A is a finite non-empty set, then $\lim_{n \in A} a_n = L$ for all sequences $\{a_n\}$ and for all real numbers L.

Answer The logical statement to be proved is:

If $\varepsilon > 0$ there is N in A so that if n is in A and $n > N$, then $|a_n - L| < \varepsilon$.

Since A is finite, there is N, the maximum integer in the non-empty finite set A, so that there is *no* n in A with $n > N$. If we use this N in the limit definition, we have an implication where the hypothesis is always false. So the implication itself is true.

b) (6 points) Suppose $A_1, A_2, \ldots, A_k$ is a pairwise disjoint decomposition of the positive integers into infinite subsets A_j with $1 \le j \le k$. That is, each of the A_j's is an infinite subset of the positive integers and each positive integer is in exactly one of the A_j's. Prove that $\lim_{n \in \mathbb{N}} a_n = L$ if and only if $\lim_{n \in A_j} a_n = L$ for all j.

Answer First, suppose $\lim_{n \in \mathbb{N}} a_n = L$. Thus, given $\varepsilon > 0$ there is N in $\mathbb{N}$ so that if $n > N$, then $|a_n - L| < \varepsilon$. But since $A_j \subset \mathbb{N}$, we know that "If n is in A_j and $n > N$, then $|a_n - L| < \varepsilon$" is true since "n is in A_j and $n > N$" implies $n > N$.

Now take some $\varepsilon > 0$. We assume that for each j with $1 \le j \le k$, there is $N_j \in A_j$ so that if n is in A_j and $n > N_j$, then $|a_n - L| < \varepsilon$. The set $\{N_1, N_2, \ldots, N_k\}$ is a finite non-empty set of positive integers and therefore has a maximum, N. We also know that $\mathbb{N} = \bigcup_{j=1}^{k} A_j$. Thus if n is any integer greater than N, n is in one of the A_j's and $n > N \ge N_j$. So $|a_n - L| < \varepsilon$ using the limit statement for that A_j. And we're done.

c) (6 points) Is a statement similar to b) true if the positive integers are written as a union of an infinite number of pairwise disjoint infinite subsets? Either prove such a statement or give a counterexample.

Answer The implication is false. Here is one counterexample. Every positive number n can be written as $n = 2^k(2j+1)$ where j and k are unique non-negative integers. For n described this way, define a_n to be $\frac{1}{k+1}$. Now let A_j for $j \ge 0$ be all the integers $\{2^k(2j+1)\}_{k \ge 0}$. The sets $\{A_j\}$ form a pairwise disjoint decomposition of the positive integers, and each A_j has infinitely many elements. Surely $\lim_{n \in A_j} a_n = 0$ since $\frac{1}{k+1} < \varepsilon$ when k is larger than any integer in A_j greater than $\frac{1}{\varepsilon}$. But $\lim_{n \in \mathbb{N}} a_n$ is not 0. We look at a specific value of ε: take $\varepsilon = 1$. Consider any positive integer N. Let n be any odd integer larger than N (there are infinitely many!). Since $a_n = 1$ for odd integer n's (for such n's, we have $k = 0$) we can't conclude that $|a_n - 0| = 1$ is less than this ε.

Comment Here's a better example, basically *copied from answers written by several students*. Use the same collection of A_j's, but define a_n to be 1 if n is a power of 2 and 0 otherwise. Then on all of $\mathbb{N}$ the sequence doesn't converge (it always has 0 and 1 terms "high up") and on each A_j the sequence converges to 0 (it is eventually equal to 0).