(20) 1. a) Suppose that a recursively defined sequence is given by $a_n = \begin{cases} 1, & \text{if } n = 1 \\ 2 + \sqrt{a_{n-1}}, & \text{if } n > 1 \end{cases}$.
Prove that $\{a_n\}$ converges and find its limit. Give supporting reasoning.

Comment This is an increasing, bounded sequence. The simple results connecting limits and arithmetic allow the limit to be found.

b) Suppose that a recursively defined sequence is given by $b_n = \begin{cases} 1, & \text{if } n = 1 \text{ or } 2 \\ b_{n-1} + \frac{1}{(b_{n-2})^2} & \text{if } n > 2 \end{cases}$.
Does $\{b_n\}$ converge? If not, why not? If it does, find its limit. Give supporting reasoning.

Comment This is also an increasing sequence, but its convergence leads to an immediate contradiction using limit/arithmetic ideas.

(20) 2. a) Suppose (X, d) is a metric space, S is a subset of X, and $w \in X$. Define "w is a limit point of S". Then consider S, a subset of $\mathbb{R}^2$, defined by $S = \{(x, y) \in \mathbb{R}^2 : x = \frac{1}{n} \text{ and } y = \frac{1}{m} \text{ for } n, m \text{ positive integers}\}$. Find all limit points of S in $\mathbb{R}^2$ with the usual metric.

Comment Many people did not write the requested definition (why?). S is from an exercise in Buck's *Advanced Calculus* book – the U.S. predecessor to our text.

b) Suppose (X, d) is a metric space, and U and C are subsets of X. Define "U is an open set" and "C is a closed set". Prove that C is closed if and only if every limit point of C is in C.

Comment Again, some people didn't write the definitions which were clearly requested. This is strange, to me. These are easy points.

(20) 3. Suppose $a < c < b$, α is an increasing function on $[a, b]$, and f is a bounded real-valued function on $[a, b]$ which is continuous on $[a, c)$ and $(c, b]$. These assumptions hold for both parts of this problem.

a) Prove that if α is continuous at c, then $f \in \mathcal{R}(\alpha)$.

Comment This is an simplified version of a result in Rudin's text, Theorem 6.10.

b) Find an example of an α which is *not* continuous at c and an f which is *not* continuous at c so that $f \notin \mathcal{R}(\alpha)$.

Comment There are simple examples – jumps at the same point for both α and f will give one.

(20) 4. True or false. If true, give a very brief explanation of why the statement is correct. If false, supply an example showing why the implication is false.

a) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous and bounded. Then f is constant.

TRUE OR FALSE? ____________

Comment $f(x) = \sin x$

b) Suppose a metric space has the property that every real-valued continuous function is bounded. Then the metric space is compact.

TRUE OR FALSE? ____________

Comment The Tietze Extension Theorem is *not* needed!

c) Every metric space is the union of a collection of open balls of finite radius which are pairwise disjoint (the intersection of any two of the balls is empty).

TRUE OR FALSE? ___________

Comment If true, metric spaces with more than 1 point would not be connected.

d) Every complete metric space is connected.

TRUE OR FALSE? ___________

Comment Nah

e) If $\{a_n\}$ is a complex sequence for which $\sum_{n=1}^{\infty} |a_n|$ converges, then $\limsup_{n\to\infty} |a_n|^{1/n} < 1$.

TRUE OR FALSE? ___________

Comment Nah

(20) 5. Suppose S^1 is the subset of $\mathbb{R}^2$ of points (x, y) which satisfy the equation $x^2 + y^2 = 1$. Here $\mathbb{R}^2$ and $\mathbb{R}$ have the usual metrics. Prove that there is no 1-1 (injective) continuous map from S^1 to $\mathbb{R}^1$.

Note The mapping is *not* required to be onto (surjective).

Comment This problem was suggested by Professor Ferry. The problem I wanted to use, a problem simple to state about arcs (continuous images of intervals) crossing in a square does not seem to have a simple solution. Two-dimensional topology is more intricate than what happens in $\mathbb{R}$! I have consulted extensively and the only solutions I know for it are not easy. The problem here can be solved with elementary methods, though.

(20) 6. Suppose $\{a_n\}$ and $\{b_n\}$ are real *positive** sequences.

a) Prove that $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

Comment A few students clearly showed that this problem, as originally stated, without *positive* or some other appropriate modifier, is either FALSE or, more kindly, difficult to make sense of. For example, consider $a_n = n$ and $b_n = -n$. Then the inequality becomes $0 \leq \infty - \infty$ and we shouldn't be doing arithmetic with both ends of the extended real numbers. That sort of thing leads to bad results. Some additional word or phrase is needed to make the inequality valid.

b) Give an example to show that $\limsup_{n\to\infty} (a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ may be false.

(20) 7. Suppose (X, d) is a metric space and S is a non-empty subset of X. If $w \in X$, define $D(w) = \inf_{x\in S} d(x, w)$.

a) Prove that $D\colon X \to \mathbb{R}$ is continuous (it is actually uniformly continuous).

Comment $\stackrel{\Delta}{\leq}$

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* Word inserted *after* the exam!

b) Prove that $D(x) = 0$ if and only if $x \in \overline{S}$ (x is in the closure of S).

Comment Too much like some of the solution of problem 2b).

c) If $X = \mathbb{R}$ with the usual metric, is the function D differentiable everywhere for every choice of S?

Comment If $S = 0$ then $D(x) = |x|$.

(20) 8. Suppose f is a continuous real-valued function on $(0, \infty)$ and define $F\colon (0, +\infty) \to \mathbb{R}$ by $F(x) = \int_1^x f\left(\frac{u^2+1}{u}\right) \frac{du}{u}$. Prove that $F\left(\frac{1}{x}\right) = -F(x)$ for all $x > 0$.

Comment This is also from Buck's book, where it is given with $e^{\frac{u^2+1}{u}}$ inside the integral, a magnificent "distracter" since the exponential function has nothing to do with the problem. I would solve this by differentiating both sides of the candidate equation, using the Fundamental Theorem of Calculus and the Chain Rule. The derivatives are equal, and both sides are equal when $x = 1$ so the functions on their connected domain are the same. A number of students found a neater solution. They changed variables in the definite integral – after reading their solutions this is "easy" – by just using, say, $v = \frac{1}{u}$, and then the equation required pops out: nice.

(20) 9. Suppose f is continuous on $[0, 1]$ and $\varepsilon > 0$. Prove that there is a piecewise linear function g on $[0, 1]$ so that $|g(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$.

Advice g is a *piecewise linear function* on $[0, 1]$ if g is continuous on $[0, 1]$ and there is a finite partition $\mathcal{P} = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ of $[0, 1]$ so that $g|_{[x_j, x_{j+1}]}$ is equal to an affine function ("$A_j x + B_j$") for all integers j from 0 to $n - 1$.

Comment Citing uniform continuity gives away most of the problem. Then use line segments to "connect the dots" of the graph of f restricted to points on a fine enough partition. This satisfies the requested assertion.

(20) 10. a) In this part of the problem, we consider sequences $\{x_n\}$ in $\mathbb{R}$ with the usual metric. Then give examples with brief explanation of:

i) A sequence in $\mathbb{R}$ which has an uncountable number of distinct subsequential limits.

Comment $\mathbb{Q}$ presented sequentially will certainly serve.

ii) A sequence in $\mathbb{R}$ which has a countably infinite number of distinct subsequential limits.

iii) A sequence in $\mathbb{R}$ which has exactly three distinct subsequential limits.

b) In this part of the problem, $g\colon \mathbb{R} \to \mathbb{R}$ and g is differentiable at 2 with $g'(2) = 4$. Show that there exists $\delta > 0$ so that if $2 < x < 2 + \delta$ then $g(x) > g(2) + 3(x - 2)$.

Comment A nasty problem because it is very straightforward and was the last part of the last problem on the exam. It is copied, with permission, from a Math 311 final exam given this semester. Those students did almost as well as this class, but they had been prepared for such a question by the instructor. Your instructor prepared you only for ... chaos. No further continuity or differentiability of g is needed, nor should any be assumed. The definition of derivative alone is sufficient.

Final Exam for Math 411

December 16, 2008

NAME ____

Do all problems, in any order.

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total Points Earned:		