(20) 1. a) Suppose that a recursively defined sequence is given by $a_n = \begin{cases} 1, & \text{if } n = 1 \\ 2 + \sqrt{a_{n-1}}, & \text{if } n > 1 \end{cases}$.
Prove that $\{a_n\}$ converges and find its limit. Give supporting reasoning.

b) Suppose that a recursively defined sequence is given by $b_n = \begin{cases} 1, & \text{if } n = 1 \text{ or } 2 \\ b_{n-1} + \frac{1}{(b_{n-2})^2} & \text{if } n > 2 \end{cases}$.
Does $\{b_n\}$ converge? If not, why not? If it does, find its limit. Give supporting reasoning.

(20) 2. a) Suppose (X, d) is a metric space, S is a subset of X, and $w \in X$. Define "w is a limit point of S". Then consider S, a subset of $\mathbb{R}^2$, defined by $S = \{(x, y) \in \mathbb{R}^2 : x = \frac{1}{n} \text{ and } y = \frac{1}{m} \text{ for } n, m \text{ positive integers}\}$. Find all limit points of S in $\mathbb{R}^2$ with the usual metric.

b) Suppose (X, d) is a metric space, and U and C are subsets of X. Define "U is an open set" and "C is a closed set". Prove that C is closed if and only if every limit point of C is in C.

(20) 3. Suppose $a < c < b$, α is an increasing function on $[a, b]$, and f is a bounded real-valued function on $[a, b]$ which is continuous on $[a, c)$ and $(c, b]$. These assumptions hold for both parts of this problem.

a) Prove that if α is continuous at c, then $f \in \mathcal{R}(\alpha)$.

b) Find an example of an α which is *not* continuous at c and an f which is *not* continuous at c so that $f \notin \mathcal{R}(\alpha)$.

(20) 4. True or false. If true, give a very brief explanation of why the statement is correct. If false, supply an example showing why the implication is false.

a) Suppose $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous and bounded. Then f is constant.

TRUE OR FALSE? ____________

b) Suppose a metric space has the property that every real-valued continuous function is bounded. Then the metric space is compact.

TRUE OR FALSE? ____________

c) Every metric space is the union of a collection of open balls of finite radius which are pairwise disjoint (the intersection of any two of the balls is empty).

TRUE OR FALSE? ____________

d) Every complete metric space is connected.

TRUE OR FALSE? ____________

e) If $\{a_n\}$ is a complex sequence for which $\sum_{n=1}^{\infty} |a_n|$ converges, then $\limsup_{n\to\infty} |a_n|^{1/n} < 1$.

TRUE OR FALSE? ____________

(20) 5. Suppose S^1 is the subset of $\mathbb{R}^2$ of points (x, y) which satisfy the equation $x^2 + y^2 = 1$. Here $\mathbb{R}^2$ and $\mathbb{R}$ have the usual metrics. Prove that there is no 1-1 (injective) continuous map from S^1 to $\mathbb{R}^1$.

Note The mapping is *not* required to be onto (surjective).

(20) 6. Suppose $\{a_n\}$ and $\{b_n\}$ are real *positive** sequences.

a) Prove that $\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

b) Give an example to show that $\limsup_{n\to\infty} (a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$ may be false.

(20) 7. Suppose (X, d) is a metric space and S is a non-empty subset of X. If $w \in X$, define $D(w) = \inf_{x \in S} d(x, w)$.

a) Prove that $D\colon X \to \mathbb{R}$ is continuous (it is actually uniformly continuous).

b) Prove that $D(x) = 0$ if and only if $x \in \overline{S}$ (x is in the closure of S).

c) If $X = \mathbb{R}$ with the usual metric, is the function D differentiable everywhere for every choice of S?

(20) 8. Suppose f is a continuous real-valued function on $(0, \infty)$ and define $F\colon (0, +\infty) \to \mathbb{R}$ by $F(x) = \int_1^x f\left(\frac{u^2+1}{u}\right) \frac{du}{u}$. Prove that $F\left(\frac{1}{x}\right) = -F(x)$ for all $x > 0$.

(20) 9. Suppose f is continuous on $[0, 1]$ and $\varepsilon > 0$. Prove that there is a piecewise linear function g on $[0, 1]$ so that $|g(x) - f(x)| < \varepsilon$ for all $x \in [0, 1]$.

Advice g is a *piecewise linear function* on $[0, 1]$ if g is continuous on $[0, 1]$ and there is a finite partition $\mathcal{P} = \{0 = x_0 < x_1 < \ldots < x_n = 1\}$ of $[0, 1]$ so that $g|_{[x_j, x_{j+1}]}$ is equal to an affine function ("$A_j x + B_j$") for all integers j from 0 to $n - 1$.

(20) 10. a) In this part of the problem, we consider sequences $\{x_n\}$ in $\mathbb{R}$ with the usual metric. Then give examples with brief explanation of:

i) A sequence in $\mathbb{R}$ which has an uncountable number of distinct subsequential limits.

ii) A sequence in $\mathbb{R}$ which has a countably infinite number of distinct subsequential limits.

iii) A sequence in $\mathbb{R}$ which has exactly three distinct subsequential limits.

b) In this part of the problem, $g\colon \mathbb{R} \to \mathbb{R}$ and g is differentiable at 2 with $g'(2) = 4$. Show that there exists $\delta > 0$ so that if $2 < x < 2 + \delta$ then $g(x) > g(2) + 3(x - 2)$.

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* Word inserted *after* the exam!

Final Exam for Math 411

December 16, 2008

NAME ____

Do all problems, in any order.

Problem Number	Possible Points	Points Earned:
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total Points Earned:		