Rudin Chapter 5, problems 22 and 23

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Suppose f is a real function on $(-\infty, \infty)$. We say x is a fixed point of f if $f(x) = x$.

a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Proof: Suppose that f satisfies the hypotheses, but, for distinct real numbers a and b with $a < b$, $f(a) = a$ and $f(b) = b$. Then, by the Mean Value Theorem, there exists $r \in (a, b)$ such that $f'(r) = \frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$, contradiction. Hence, f has at most one fixed point.

b) Show that the function defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

First, $f'(t) = 1 - e^t(1 + e^t)^{-2}$. To see that $f'(t)$ is bounded between 0 and 1, consider the following implications:

$$\begin{aligned} 0 < e^t \Rightarrow 1 < 1 + e^t \Rightarrow e^t < (1 + e^t)^2 \Rightarrow -1 < -e^t(1 + e^t)^{-2} < 0 \\ \Rightarrow 0 < f'(t) < 1. \end{aligned}$$

If f did have a fixed point, then $f(t) = t \Rightarrow (1 + e^t)^{-1} = 0$, which is a contradiction since the left-hand side is positive for all t.

This shows that boundedness of f' less than 1 does not guarantee a fixed point.

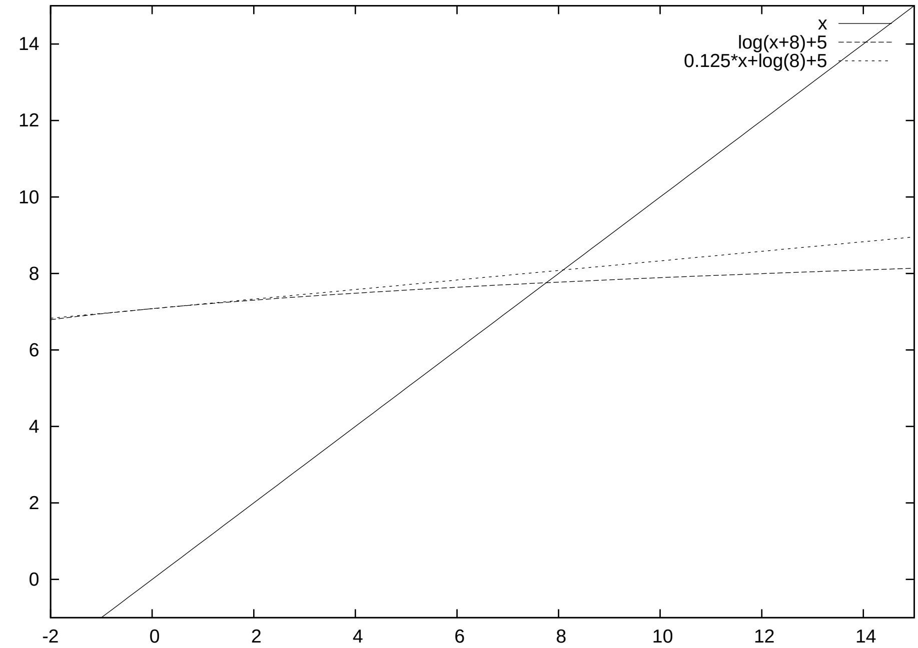
c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, then a fixed point x of f exists, and $x = \lim x_n$, where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$.

Lemma: If, in addition to the above hypotheses, $f(t) > t$ for all t, then $f(t) \leq g(t) = At + f(0)$ for all $t \geq 0$. Similarly, if $f(t) < t$ for all t, then $f(t) \geq h(t) = At + f(0)$ for all $t \leq 0$.

Proof of lemma: Note that $f(0) = g(0)$, and that g is a differentiable function with a constant derivative, namely A. Let $y \in (0, \infty)$ be arbitrary. Then, by the Mean Value Theorem, there exists $r \in (0, y)$ such that $f'(r) = \frac{f(y)-f(0)}{y}$. Since $f'(r) \leq g'(r) = A$, we have $\frac{f(y)-f(0)}{y} \leq \frac{g(y)-g(0)}{y} \Rightarrow f(y) \leq g(y)$. Hence, $f(t) \leq g(t)$ for all $t \geq 0$.

Take an arbitrary $z \in (0, \infty)$. Then, by the Mean Value Theorem, there is a point $s \in (-z, 0)$ such that $f'(s) = \frac{f(0)-f(-z)}{z}$. As above, we find that $\frac{f(0)-f(-z)}{z} \leq \frac{h(0)-h(-z)}{z} \Rightarrow -f(-z) \leq -h(-z) \Rightarrow h(-z) \leq f(-z)$. Since z was arbitrarily chosen, $h(t) \leq f(t)$ for all $t \leq 0$.

This is an image of the function $f(t) = \log(t + 8) + 5$, for which $f'(t) \leq 1/8$ for $t \geq 0$. Clearly, it cannot always be bounded between t and $(1/8)t + f(0)$.



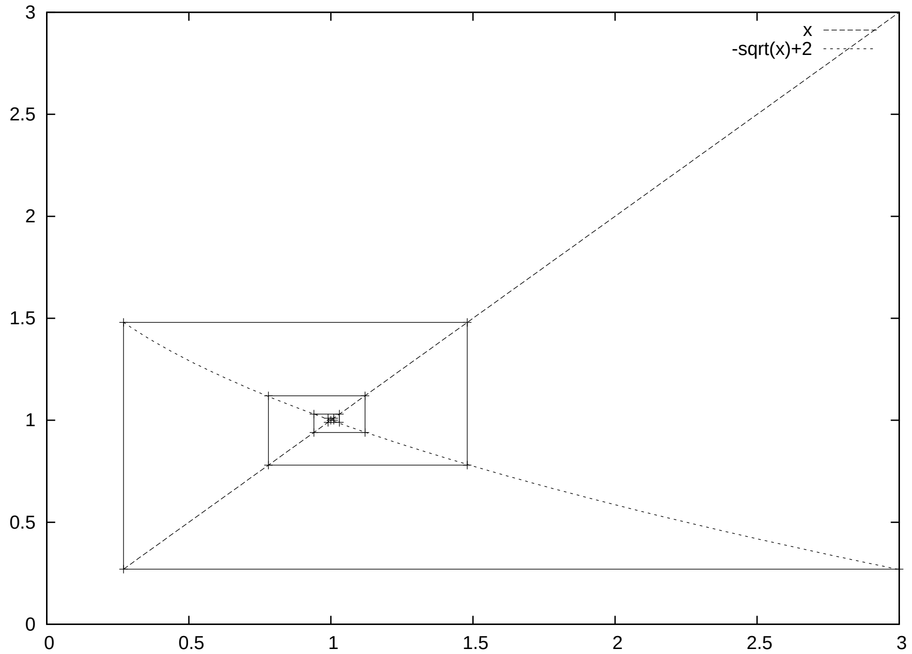
Proof of c): Recall that f can have at most one fixed point. Suppose f has none. Then, for all t, $f(t) > t$ or $f(t) < t$. If it is always the case that $f(t) > t$, then by the lemma we can conclude that $t < At + f(0)$ for all positive t; this is a contradiction, since $t = At + f(0)$ at $t = \frac{f(0)}{1-A}$. If $f(t) < t$ for all t, we have by the lemma that $At + f(0) < t$ for all negative t; again, this is a contradiction.

To see the following argument more clearly, we define $g(t) = f(t) - t$. We have assumed that $g(t) \neq 0$ for all t, and that neither $g(t) > 0$ or $g(t) < 0$ can hold for all t. Hence, there exist a and b with $a < b$ such that $g(a) > 0$ and $g(b) < 0$. By the Intermediate Value Theorem, there exists $r \in (a, b)$ such that $g(r) = 0$, contradicting that $g(t) \neq 0$ for all t. Hence, there is some point x such that $f(x) = x$.

Finally, we show that (x_n) converges to x. First, note that $|x - x_2| = |f(x) - f(x_1)|$. Then, by the Mean Value Theorem, there is some number r

between x and x_1 such that $|f'(r)| = \frac{|f(x)-f(x_1)|}{|x-x_1|}$. As $|f'(r)| \leq A$, $|x - x_2| = |f(x) - f(x_1)| \leq A|x - x_1|$. Suppose that for some natural number $n \geq 1$ that $|x - x_{n+1}| = |f(x) - f(x_n)| \leq A^n|x - x_1|$. Then, applying the Mean Value Theorem as we did in the case $n = 1$, we find that $|x - x_{n+2}| = |f(x) - f(x_{n+1})| \leq A|x - x_{n+1}| \leq A \cdot A^n|x - x_1| = A^{n+1}|x - x_1|$. So, for every n, $|x - x_n| \leq A^{n-1}|x - x_1|$. By Bernoulli's inequality, $0 \leq \lim_{n\to\infty} |x - x_n| \leq \lim_{n\to\infty} A^{n-1}|x - x_1| = 0$, and therefore $x_n \to x$.

d) Here is a picture of the algorithm converging. The path $(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to ...$ is represented by the zig-zag lines. The function whose fixed point is being found is $-\sqrt{x} + 2$, and I chose $x_1 = 3$.



As an example of how this can be applied, consider the function

$$f(x) = \frac{x^3 + 1}{3}$$

which has three fixed points. The fixed points α, β, and γ satisfy

$$-2 < \alpha < -1, \qquad 0 < \beta < 1, \qquad 1 < \gamma < 2$$

. Suppose that we have a sequence defined as in 22c.

a) If $x_1 < \alpha$, then $x_n \to -\infty$ as $n \to \infty$.

If $x_n < \alpha$, then $x_{n+1} = \frac{x_n^3+1}{3} < \alpha$. Hence, $x_n \in (-\infty, \alpha)$ for all n. It follows that $f'(x_n) > f'(\alpha) = \alpha^2 > 1$ for each n. By the Mean Value Theorem, for each n there exists a point $c_n \in (x_n, \alpha)$ such that $f'(c_n) = \frac{f(\alpha)-f(x_n)}{\alpha - x_n}$.

Hence, $|\alpha - x_{n+1}| = |f(\alpha) - f(x_n)| = f'(c_n)|\alpha - x_n| > \alpha^2|\alpha - x_n|$. Since $|\alpha - x_2| > \alpha^2|\alpha - x_1|$, the relation $|\alpha - x_n| > \alpha^{2(n-1)}|\alpha - x_1|$ holds for all $n \geq 2$.

Since $\lim_{n\to\infty} \alpha^{2(n-1)}|\alpha - x_1| = \infty$, $\lim_{n\to\infty} |\alpha - x_n| = \infty$. Thus, $x_n \to -\infty$.

b) If $\alpha < x_1 < \gamma$, then $x_n \to \beta$ as $n \to \infty$. [Note: $f'(x) = x^2$.]

The cases we consider are $x_1 \in (\alpha, -1)$, $x_1 \in [-1, -1/2)$, $x_1 \in [-1/2, 1/2]$, $x_1 \in (1/2, 1]$, and $x_1 \in (1, \gamma)$.

Case 1: First, suppose $x_1 \in [-1/2, 1/2]$. For $t \in [-1/2, 1/2]$, $f'(t) \leq 1/4 < 1$. Then, $|\beta - x_2| = |f(\beta) - f(x_1)| \leq (1/4)|\beta - x_1|$. Since x_2 is closer to β than x_1, $x_2 \in [-1/2, 1/2]$. It can be shown by an argument similar to the one given in 22c that $x_n \to \beta$.

Case 2: Next, suppose that $x_1 \in [-1, -1/2)$. Since $-1 \leq x_1 < -1/2$, $-1 \leq x_1^3 < (-1/2)^3 \Rightarrow 0 \leq \frac{x_1^3+1}{3} = x_2 < 7/24 < 1/2$. Thus, $x_2 \in [-1/2, 1/2]$, and it follows from case 1 that $x_n \to \beta$.

Case 3: It is easy to see that if $x_n \in (\alpha, -1)$ then $x_{n+1} < 0$, and therefore $x_{n+1} \notin (1/2, \gamma)$. Thus, $x_{n+1} \in [-1, 1/2]$, in which case convergence to β follows, or $x_{n+1} \in (\alpha, -1)$.

So, suppose that $x_n \in (\alpha, -1)$ for all n. If $x_1 \in (\alpha, -1)$, then $\alpha < x_1 \Rightarrow \alpha < \frac{x_1^3+1}{3} = x_2$. An inductive argument shows that $x_n > \alpha$ for all n.

Notice that for $t \in (\alpha, -1)$ we have that $f'(t) > 1$. Consider $|\alpha - x_{n+1}| = |f(\alpha) - f(x_n)|$. By the Mean Value Theorem, there exists $c_n \in (\alpha, x_n)$ such that $f'(c_n) = \frac{f(\alpha)-f(x_n)}{\alpha - x_n}$. Thus, $|\alpha - x_{n+1}| > |\alpha - x_n|$ for all n. Therefore, (x_n) is monotonically increasing, and is bounded above by -1. Hence, (x_n) must converge to some number $x \in (\alpha, -1)$. From continuity of f and the fact that (x_n) and $(f(x_n))$ have the same limit, it follows that $f(x) = x$, so x is a fixed point. But this is a contradiction, since there is no fixed point in $(\alpha, -1)$. Hence, the sequence is not bounded above by -1, and therefore $x_n \to \beta$.

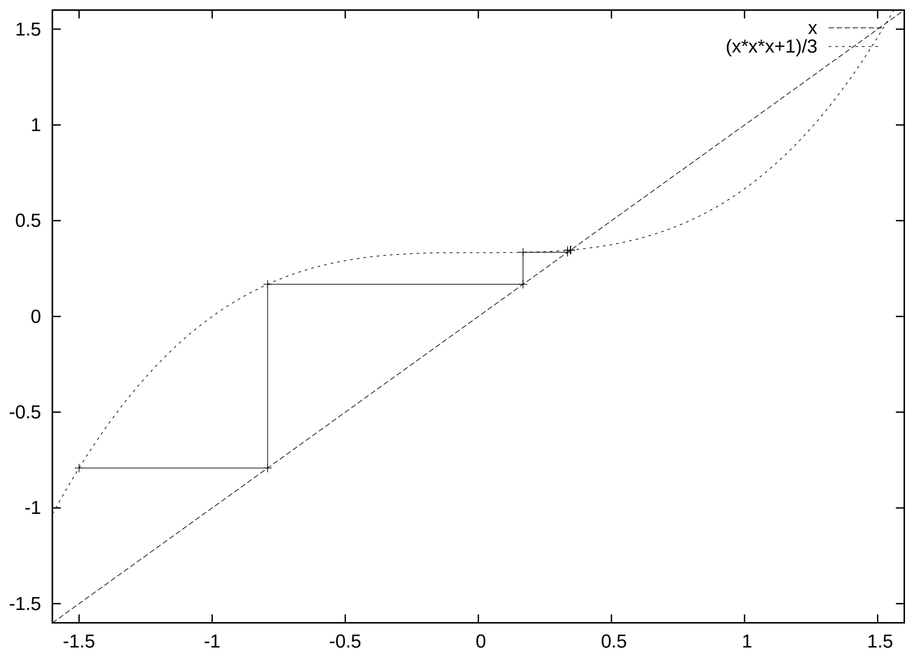
Case 4: The argument for when $x_1 \in (1/2, 1]$ is similar to case 2.

Case 5: The argument for when $x_1 \in (1, \gamma)$ is similar to case 3.

c) If $\gamma < x_1$, then $x_n \to \infty$ as $n \to \infty$.

The result follows from the same argument used in part a).

Here is an image of the fixed point iteration algorithm converging to β. I chose $x_1 = -3/2$.


x
(x*x*x+1)/3
1.5
1
0.5
0
-0.5
-1
-1.5
-1.5
-1
-0.5
0
0.5
1
1.5