Math 411 **Homework #4** **10/13/2008**

Due at the beginning of class, Thursday, October 23, 2008*

1. (10 points) Suppose that $\{x_n\}$ is a sequence in a metric space, (X, d). A *quadratically increasing subsequence* (called *QIS* in this problem) is the following: if $\{x_n\}$ is a sequence corresponding to the mapping $f\colon \mathbb{N} \to X$, then $g\colon \mathbb{N} \to X$ is a QIS of f if $g = f \circ I$ where there is $A > 0$ and $I\colon \mathbb{N} \to \mathbb{N}$ is a strictly increasing mapping satisfying $I(n) \geq An^2$ for all n. Prove that if $\{x_n\}$ is a sequence in a metric space and if all QIS's of $\{x_n\}$ converge (the limit may depend on the subsequence, of course), then $\{x_n\}$ converges.

2. (10 points) Following the definition of Cauchy sequence in class, we showed that there is a sequence, $\{x_n\}$, in $\mathbb{R}$ which satisfies $d(x_n, x_{n+1}) < \frac{1}{n}$ and which does *not* converge (so it also can't be a Cauchy sequence).

a) Prove the following: if $n > 10^{10}$, then $\frac{1}{n} < \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

b) Suppose $\{x_n\}$ is a sequence in a complete metric space (X, d) which satisfies the following estimate: $d(x_n, x_m) < \frac{1}{\sqrt{n}+\sqrt{m}}$. Prove that $\{x_n\}$ converges.

c) Observe that a sequence which satisfies the inequality in b) must also satisfy the inequality $d(x_n, x_{n+1}) < \frac{1}{\sqrt{n}+\sqrt{n+1}}$. In view of the example discussed in class and the estimate verified in a), the example discussed in class satisfies a much more stringent (stricter!) inequality. Why does that example *not* converge?

Textbook problems Chapter 3: 14 (a), (b), (c), and (d) (*not* (e)), 20, 21, and 22. Each is worth 10 points.
Problem 14 is an example of a famous theme of analysis, combinatorics, and probability. I thought that part (e) would be too much, but you can look at it.

Math 411 **The first exam** **10/13/2008**

The first exam will take place at the standard class time and place on Monday, October 20, 2008. The exam will cover material we've discussed since the beginning of the semester. The part of the textbook subject to examination is about two and a half chapters, until p. 57, **Some Special Sequences**. Notice, though, that during the discussion of the Archimedean property we did analyze the sequences $\left\{\frac{1}{n}\right\}$ and $\{x^n\}$. Behavior of the other "special sequences" will not be tested.
Professor Teixeira's first exam and review problems (there's a link on our course home page) are *mostly* suitable for you to consider, as are our own homework problems (including this assignment!). Some of his problems involve material we have not touched on, however.

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* Unless renegotiated.