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Mathematical Analysis I
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*On the Countability of Algebraic Numbers and the Existence of Transcendental Numbers*

**Definition.** *A complex number $z$ is said to be* ***algebraic*** *if it satisfies some polynomial equation of positive degree*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

*with integer coefficients $a_i$ not all equal to 0.*
*A complex number which is not algebraic is said to be* ***transcendental.***

**Theorem.** *The algebraic numbers are countable.*

*Proof.* Let $P_k$ be the set of all $k^{th}$ degree polynomials with integer coefficients. Observe that for some polynomial $p \in P_k$, p is defined uniquely by its $k+1$ coefficients. These coefficients can be taken from a $(k+1)$-tuple of $\mathbb{Z}^{k+1}$, which is a countable set. Thus $P_k$ is countable.

For $p \in P_k$, let $R_p := \{z \in \mathbb{C} | p(z) = 0\}$ be the set of all roots of $p$. It is intuitively obvious to the most casual of observers that $R_p$ is finite. *A polynomial of degree $k$ has at most $k$ roots).* Then the set of all roots of polynomials of degree $k$ can be defined as $W_k = \bigcup_{p \in P_k} R_p$. This is a countable union of countable sets, and so $W_k$ is countable.

Let $\mathscr{A}$ be the set of all algebraic numbers, defined by $\mathscr{A} = \bigcup_{k \in \mathbb{N}} W_k$. This is (yet again) a countable union of countable sets. □

**Corollary.** *The transcendental numbers are uncountable.*

*Proof.* Let $\mathscr{T}$ be the set of all transcendental numbers and let $\mathscr{A}$ be the set of all algebraic numbers. By definition, $\mathbb{R} \setminus \mathscr{A} = \mathscr{T}$. Then $\{\mathscr{A}, \mathscr{T}\}$ is a partition of the real numbers. Assume there are countably many transcendental numbers. Then $\mathscr{A} \cup \mathscr{T}$ is countable. But $\mathscr{A} \cup \mathscr{T} = \mathbb{R}$, an uncountable set. From this contradiction, we have that the transcendental numbers are uncountable. □