Math 411 9/1/2008

Not a reading list (just a discussion of some books)

The textbook for the course is *Principals of Mathematical Analysis* (third edition, $143*). *Principals* is also called “baby Rudin” since the author has written other texts on more advanced subjects. *Principals* is the conventional text for beginning professional analysis courses, at most schools both in the U.S. and internationally. It essentially describes the basic language for analysis, a vast area of mathematics, using style and vocabulary which became canonical about a half century ago. The proofs are almost all very clever, the expository style is minimalist, and there are *no* pictures. It can be difficult to believe that this austere bible describes a subject created by human beings.

• Calculus

Calculus is the basis of analysis. Here are some English-language texts which would be appropriate background for this course. They feature rigorous discussions of calculus. *A Course of Pure Mathematics* ($33) by G. H. Hardy is a century old. This lovely book was written by one of the great mathematicians of that time. *Calculus* ($85) by Michael Spivak is used in a number of honors calculus courses. It covers one variable calculus. Tom M. Apostol’s *Calculus* is in two volumes. The first ($161) covers one variable calculus and the second ($129) discusses several variable calculus.

• History

Some historical perspective about analysis may be useful, especially to students paralyzed by the appearance of perfection in Rudin’s book. Two fairly recent books by David Bressoud, *A Radical Approach to Real Analysis*, Second Edition ($53) and *A Radical Approach to Lebesgue’s Theory of Integration* ($33), contain thorough discussions of how the great mathematical minds of the nineteenth and early twentieth centuries created the structure of analysis which is so elegantly described in *Principals*. In particular, some of the original examples are given, and the intricate variety of approaches to definitions and theorems and proofs are explored. The exposition is excellent.

Some of the original sources in the subject are very readable. If you are interested in logic and the foundations of mathematics, Dedekind’s original essays are remarkably rewarding. They can be read in the original German, of course, and also in English translation in *Essays on the Theory of Numbers*, ($9, Dover). They aren’t about number theory. The book has translations of two extended essays: *Stetigkeit und irrationale Zahlen* (“Continuity and irrational numbers” – a discussion of Dedekind cuts) and *Was sind und was sollen die Zahlen?* (approximately, “What are numbers and what are they about?” – going from basic set theory to the Peano axioms for the positive integers). There’s a link to a nice discussion of the second essay at `http://aleph0.clarku.edu/~djoyce/numbers/`.

• Heresies

Here are brief discussions and references to two other ways of “doing” analysis. Historically, these have been losers (?) compared to the orthodox view we’ll study in Math 411. But over a long time, both of these approaches have returned repeatedly to the attention of mathematicians who become dissatisfied with the usual methods.

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* The prices quoted are from `Amazon`. There are likely cheaper alternate sources, but this one is easy to find and gives a simple way to compare costs.

Greatly simplifying, in **constructive analysis** real numbers are described as sequences of rationals with predicted mutual differences. This is an old idea (lasting more than a century), and was most recently championed by an eminent and accomplished classical analyst, Errett Bishop, who spent the last half of his career showing that much of modern mathematics can be done using this approach. Constructive analysis could be understood as an abstract version of *algorithmic* analysis. One weird and wonderful consequence is that some uses of the law of the excluded middle are essentially not allowed. To prove that something is POSITIVE you need to show that it is bigger than a positive number, not that it is neither negative nor zero (those assertions don't verify *positivity*). On a more advanced level, existence statements verified with the Axiom of Choice can't be used and similar results are verified with approximation schemes. The Axiom of Choice is used widely in advanced mathematics (simplest examples involve asserting the existence of maximal ideals in unitary rings, and the existence of bases for all vector spaces) so to get corresponding *constructive* results, careful approximation ideas must be used. Sometimes quite interesting developments occur. Bishop appears as a coauthor in *Constructive Analysis* by Bishop and Bridges (now only available used, $200, but in the library). There's a more recent text, *Real Analysis: A Constructive Approach* ($100), written by Mark Bridger.

A totally different approach is **non-standard analysis**, an effort to establish a coherent framework for the idea that very very small (o.k., *infinitesimal*) numbers really do exist, and so do their reciprocals (infinitely large numbers). *Elementary Calculus* (1976) by H. Jerome Keisler is a basic calculus book with a non-standard approach. It is out of print but is available for free online at `http://www.math.wisc.edu/~keisler/calc.html`. Also look at *Infinitesimal Calculus* ($11, Dover) by Henle and Kleinberg. A more complete treatment is in *Non-standard Analysis* ($42) by Abraham Robinson. In non-standard analysis, the real numbers, $\mathbb{R}$, are replaced by various different models, enlargements of the reals, $\mathbb{R}^*$, which allow epsilons, positive numbers smaller than $\frac{1}{n}$ for all positive integer n's, to exist. An important idea, the Transfer Principle, is used to take certain logical statements about $\mathbb{R}$ and change them into equally true logical statements about $\mathbb{R}^*$. Since many analysts might believe in infinitesimals anyway (!), this may not be startling. There are novel non-standard proofs of standard theorems and there have even been, I believe, some new analysis theorems first proved using non-standard methods.

• Alternative texts and an additional reference

Advanced Calculus ($62) by Buck was probably the text most widely used in the United States before the hegemony of *Principals* was established. It is interesting to compare the theorems and discussions in Buck's book, which *has pictures*, with those of Rudin. A recent competitor is *The Way of Analysis* ($99) by Robert Strichartz. His text covers about the same material as *Principals*, but there's an effort to be more "reader friendly" – not every proof is awesomely terse, and the whole approach is less austere.

The book *Counterexamples in Analysis* ($10, Dover) by Gelbaum and Olmstead is *very useful**. It was the first in a series of "counterexamples" books. The Dover reprint is cheap. Buy it.

* When a family member was taking a *Principals* course at another school, my own copy of this book mysteriously disappeared.