Fix $b > 1$

Our goals are to define b^x for all $x \in R$, and to verify that our favorite laws of exponents are true with this definition.

- $b^x b^y = b^{x+y}$
- $(b^x)^y = b^{xy}$
- $a^x b^x = (ab)^x$

We will assume that these laws hold for $x, y \in \mathbb{Z}$.

Less ambitious goals: Define b^r for all $r \in \mathbb{Q}$, and verify the laws of exponents with this definition.

Lemma 1. *If* $m, n \in \mathbb{Z}$, $\sqrt[n]{\sqrt[m]{b}} = \sqrt[nm]{b}$

Proof.

$$\begin{aligned} a &= \sqrt[n]{\sqrt[m]{b}} \\ a^n &= \sqrt[m]{b} \\ (a^n)^m &= b \\ a^{nm} &= b \\ a &= \sqrt[nm]{b} \end{aligned}$$

□

If $r = \frac{m}{n}$, and $m, n \in \mathbb{Z}$, then we would like to define

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$

However, for this definition to be valid, we need to show that if $\frac{m}{n} = \frac{p}{q}$, then $\sqrt[n]{b^m} = \sqrt[q]{b^p}$.

Proposition 1. *If* $m, n, p, q \in \mathbb{Z}$, $n, q > 0$, *and* $mq = np$, *then* $\sqrt[n]{b^m} = \sqrt[q]{b^p}$

Proof.

$$\begin{aligned} mq &= np \\ b^{mq} &= b^{np} \\ \sqrt[nq]{b^{mq}} &= \sqrt[nq]{b^{np}} \\ \sqrt[n]{\sqrt[q]{(b^m)^q}} &= \sqrt[q]{\sqrt[n]{(b^p)^n}} \\ \sqrt[n]{b^m} &= \sqrt[q]{b^p} \end{aligned}$$

□

So now we can define b^r for $r \in \mathbb{Q}$ as we wanted.
Now we would like for our favorite exponent laws to be true for rationals. The proofs of these are all very similar, so we will only show one.

Proposition 2. *If $m, n, p, q \in \mathbb{Z}$, $n, q > 0$, then $b^{\frac{m}{n}} b^{\frac{p}{q}} = b^{\frac{m}{n} + \frac{p}{q}}$*

Proof.

$$\begin{aligned} a &= b^{\frac{m}{n}} b^{\frac{p}{q}} \\ a^{nq} &= (b^{\frac{m}{n}} b^{\frac{p}{q}})^{nq} \\ a^{nq} &= (b^{\frac{m}{n}})^{nq} (b^{\frac{p}{q}})^{nq} \\ a^{nq} &= (b^m)^q (b^p)^n \\ a^{nq} &= b^{mq} b^{pn} \\ a^{nq} &= b^{mq+pn} \\ a &= b^{\frac{mq+pn}{nq}} \\ a &= b^{\frac{m}{n} + \frac{p}{q}} \end{aligned}$$

□

We have now completed our less ambitious goal, and we will try to achieve our original goal.
Consider the set

$$B(x) = \{b^{\tilde{x}} | \tilde{x} \in \mathbb{Q} \text{ and } \tilde{x} \leq x\}$$

$B(x)$ is not empty because we can always find an integer smaller than a fixed real. $B(x)$ is bounded above because we can always find an integer bigger than a fixed real.

Proposition 3. *If $r \in \mathbb{Q}$, then* $\sup B(r) = b^r$

Proof.
Step 1: If $b^{\tilde{r}} \in B(r)$
then $\tilde{r} \leq r$
so $b^{\tilde{r}} \leq b^r$.
so b^r is *an* upper bound of $B(r)$.
Step 2: b^r is an element of $B(r)$,
so if a is an upper bound of $B(r)$, then $b^r \leq a$.
Therefore, $b^r = \sup B(r)$ □

Now we can define $b^x = \sup B(x)$ for all $x \in \mathbb{R}$ and we know that this definition agrees with the definition we made earlier for rationals.
Note that this would work if we had used strict equality in our definition of $B(x)$.

Claim 1. *For any $x \in \mathbb{R}$,*

$$\sup\{b^s | s \in \mathbb{Q}, s < x\} = \sup\{b^s | s \in \mathbb{Q}, s \leq x\}$$

Proof. Let S be the left hand side above, and b^x be the right hand side.
It is clear that $S \leq b^x$
It is clear that if $x \notin \mathbb{Q}$ then $S = b^x$. So let us assume, for contradiction, that $x \in \mathbb{Q}$ and $S < b^x$.
Consider $\frac{S}{b^x} < 1$. I claim that there is $n \in \mathbb{N}$ with $\frac{S}{b^x} < b^{-\frac{1}{n}} < 1$.
If this is true, then $S < b^{x-\frac{1}{n}}$, but $x - \frac{1}{n} \in \mathbb{Q}$, so we have just found an element of our LHS set that is greater than S, so we have a contradiction. □

Claim 2. *(used in previous proof with $a = \frac{S}{b^x}$) If $a < 1$, then there is $n \in \mathbb{N}$ with $a < b^{-\frac{1}{n}}$*

Proof. we can find an n to satisfy $(\frac{1}{a})^n > b$ from the corollary proved earlier by Professor Greenfield. Then

$$\begin{aligned} \left(\frac{1}{a}\right)^n &> b \\ a^n &< b^{-1} \\ a &< b^{-\frac{1}{n}} \end{aligned}$$

□

Now we would like to show that $b^{x+y} = b^x b^y$. However, this is by far the hardest part of the presentation, so instead, we'll prove some completely irrelevant lemmas.

Lemma 2. *Let $A, B \subset \mathbb{R}_+$ and A, B are bounded above.*
Let $C = \{ab | a \in A, b \in B\}$.
Then $\sup A \sup B = \sup C$.

Proof. Let $ab \in C$.
Then $ab \leq \sup A \sup B$ so $\sup A \sup B$ is an upper bound for C.

Now we must show that if z is an upper bound of C, then $\sup A \sup B \leq z$.
Let $a \in A$, $b \in B$ Then

$$\begin{aligned} z &\geq ab \\ \frac{z}{a} &\geq b \end{aligned}$$

So $\frac{z}{a}$ is an upper bound for B. This means

$$\frac{z}{a} \geq \sup B$$

This can be rearranged to

$$\frac{z}{\sup B} \geq a$$

so

$$\frac{z}{\sup B} \geq \sup A$$

and from here we get that $z \geq \sup A \sup B$ as desired.
Thus $\sup A \sup B = \sup C$. □

Lemma 3. *If $\tilde{z} \in \mathbb{Q}$, $x, y \in \mathbb{R}$, $\tilde{z} < x + y$, Then there exist $\tilde{x}, \tilde{y} \in \mathbb{Q}$ with $\tilde{x} < x$ and $\tilde{y} < y$ such that $\tilde{z} = \tilde{x} + \tilde{y}$*

Proof. This is **trivial**
For any $N \in \mathbb{N}$, we can say $N\tilde{z} < Nx + Ny$,
and the bigger the N, the bigger the difference between the two sides.
Then choose a big enough N so that $N\tilde{z} < Nx + Ny - 56$
Then $N\tilde{z} < \lfloor Nx \rfloor + \lfloor Ny \rfloor$
So call those floors I_1 and I_2 respectively, to emphasize that they are integers.
We have that $I_1 \leq Nx$ and $I_2 \leq Ny$
So then $\tilde{x} = \frac{I_1}{N} \leq x$ and $\tilde{y} = \frac{I_2}{N} \leq y$
SO

$$\begin{aligned} N\tilde{z} &< I_1 + I_2 \\ \tilde{z} &< \frac{I_1}{N} + \frac{I_2}{N} = \tilde{x} + \tilde{y} \end{aligned}$$

Now we have that $\tilde{z} \leq \tilde{x} + \tilde{y}$, but we want equality. However, we can just reduce $\tilde{x}$ to be the right size. Then our new $\tilde{x}$ will be less than our old one, so its certainly less than x, and it will still be rational because its $\tilde{z} - \tilde{y}$, the difference of two rationals.

□

Proposition 4. *If* $x, y \in \mathbb{R}$, *then* $b^{x+y} = b^x b^y$.

Proof. Let $b^{\tilde{x}} \in B(x)$, $b^{\tilde{y}} \in B(y)$
So then

$$\begin{aligned} b^{x+y} = \sup B(x+y) &\geq b^{\tilde{x}+\tilde{y}} = b^{\tilde{x}} b^{\tilde{y}} \\ \sup B(x+y) &\geq \sup\{b^{\tilde{x}} b^{\tilde{y}} | b^{\tilde{x}} \in B(x), b^{\tilde{y}} \in B(y)\} \\ \sup B(x+y) &\geq \sup B(x) \sup B(y) \end{aligned}$$

To show the other direction, let $b^{\tilde{z}} \in B(x+y)$ Then $\tilde{z} < x+y$
So $\tilde{z} = \tilde{x} + \tilde{y}$ for some $\tilde{x}, \tilde{y} \in \mathbb{Q}$, and $\tilde{x} < x$, $\tilde{y} < y$.
Then

$$\begin{aligned} b^x b^y = \sup B(x) \sup B(y) &\geq b^{\tilde{x}} b^{\tilde{y}} = b^{\tilde{x}+\tilde{y}} = b^{\tilde{z}} \\ \sup B(x) \sup B(y) &\geq \sup B(x+y) \end{aligned}$$

Which finally gives us

$$\begin{aligned} \sup B(x) \sup B(y) &= \sup B(x+y) \\ b^x b^y &= b^{x+y} \end{aligned}$$

as desired □