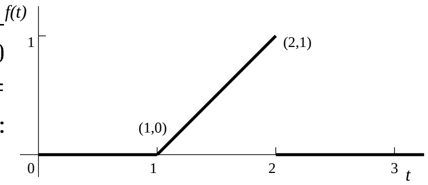
640:421:02 **Answers to the First Exam** **2/25/2004**

(16) 1. Here is a graph of the function $f(t)$ which is piecewise linear.



a) Use the definition of the Laplace transform to find the Laplace transform of the function $f(t)$. **Answer** $f(t)$ is $t-1$ for $1<t<2$ and 0 otherwise. The Laplace transform is $\mathcal{L}(f(t))(s)=\int_0^\infty e^{-st}f(t)\,dt = \int_1^2 e^{-st}(t-1)\,dt = \int_1^2 e^{-st}t\,dt - \int_1^2 e^{-st}\,dt$. The second integral is easy: $\int_1^2 e^{-st}\,dt = -\left.\frac{e^{-st}}{s}\right]_{t=1}^{t=2} = -\frac{e^{-2s}}{s}+\frac{e^{-s}}{s}$. Integrate by parts in the first integral: $\left.\begin{matrix} u=t \\ dv=e^{-st}dt\end{matrix}\right\}\left\{\begin{matrix} du=dt \\ v=-\frac{1}{s}e^{-st}\end{matrix}\right.$. Then $\int_1^2 e^{-st}t\,dt = t\left(-\frac{1}{s}e^{-st}\right)\Big]_{t=1}^{t=2} - \int_1^2 -\frac{1}{s}e^{-st}\,dt = -\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s}+\left(-\frac{1}{s^2}e^{-st}\right)\Big]_{t=1}^{t=2} = -\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}$. There are three minus signs in $-\frac{e^{-2s}}{s^2}$, two from antidifferentiation and one from integration by parts. The Laplace transform is $-\frac{e^{-2s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}$ because the second integral is subtracted.

b) Certainly $\int_0^\infty f(t)\,dt=\frac{1}{2}$. Use l'Hopital's rule to verify that $\lim\limits_{s\to 0^+} F(s)=\frac{1}{2}$. Be sure to indicate why l'Hopital's rule applies each time you use it.

Answer $F(s)=\frac{-se^{-2s}-e^{-2s}+e^{-s}}{s^2}$. When $s=0$, this is $\frac{0}{0}$. So (l'H) the limit is the same as the limit as $s\to 0^+$ of $\frac{-e^{-2s}+2se^{-2s}+2e^{-2s}-e^{-s}}{2s} = \frac{2se^{-2s}+e^{-2s}-e^{-s}}{2s}$. Again, when $s=0$, this is $\frac{0}{0}$. l'H says consider the limit as $s\to 0^+$ of $\frac{2e^{-2s}-4se^{-2s}-2e^{-2s}+e^{-s}}{2}$. This is $\frac{1}{2}$.

(14) 2. a) Use the Laplace transform to solve the initial value problem $y''-3y'=1$ with $\left\{\begin{matrix} y(0)=1 \\ y'(0)=-1\end{matrix}\right.$.

Answer The Laplace transform of y'' is $s^2Y(s)-sy(0)-y'(0)=s^2Y(s)-s+1$ and the Laplace transform of $y'=sY(s)-y(0)=sY(s)-1$. The equation becomes $s^2Y(s)-s+1-3(sY(s)-1)=\frac{1}{s}$ which then becomes $Y(s)=\frac{\frac{1}{s}+s-4}{s^2-3s}$. The rational function on the right-hand side is $\frac{s^2-4s+1}{s^2(s-3)}$. The table doesn't have an entry for this, so we'll use partial fractions: $\frac{s^2-4s+1}{s^2(s-3)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-3}=\frac{As(s-3)+B(s-3)+Cs^2}{s^2(s-3)}$. Therefore we need A and B and C so that $s^2-4s+1=As(s-3)+B(s-3)+Cs^2$. If $s=0$ then $B=-\frac{1}{3}$. If $s=3$ then $9C=-2$ and $C=-\frac{2}{9}$. Use your favorite method (mine: compare s^2 coefficients) to get $A=\frac{11}{9}$. Now we should find the inverse Laplace transform of $\frac{\frac{11}{9}}{s}+\frac{-\frac{1}{3}}{s^2}+\frac{-\frac{2}{9}}{s-3}$. We use the table to read off the answer: $y(t)=\frac{11}{9}-\frac{1}{3}t-\frac{2}{9}e^{3t}$.

b) Check that your answer satisfies the initial conditions.

Answer $y(0)=\frac{11}{9}-\frac{2}{9}e^0=1$. $y'(t)=-\frac{1}{3}-\frac{2}{3}e^{3t}$ so $y'(0)=-\frac{1}{3}-\frac{2}{3}e^0=-1$.

(12) 3. Find the Laplace transform of $H(t-3)\left(t^2+e^{5t}+1\right)$.

Answer We write $t^2+e^{5t}+1$ as a function of $t-3$ to use the shifting theorem: $t^2+e^{5t}+1=4\big((t-3)+3\big)^2+e^{5\left((t-3)+3\right)}+1=\big((t-3)^2+6(t-3)+9\big)+e^{15}e^{5(t-3)}+1$. Now we can use the table (shifting and other facts) to see that the Laplace transform of $H(t-3)\left(t^2+e^{5t}+1\right)$ is $e^{-3s}(\frac{2}{s^3}+\frac{6}{s^2}+\frac{9}{s}+e^{15}\frac{1}{s-5}+\frac{1}{s})$.

(8) 4. Compute the convolution of e^{5t} and e^{8t}.

Answer The product of the Laplace transforms is the Laplace transform of the convolution, so the answer is the inverse Laplace transform of $\frac{1}{(s-5)(s-8)}$. We can read this off the table (here $a=5$ and $b=8$): $\frac{e^{5t}-e^{8t}}{5-8}$. You could also compute the convolution from the definition: $\int_0^t e^{5(t-\tau)}e^{8\tau}d\tau = e^{5t}\int_0^t e^{3\tau}d\tau = e^{5t}\frac{1}{3}e^{3\tau}]_{\tau=0}^{\tau=t} = e^{5t}\frac{1}{3}(e^{3t}-1)$, the same answer.

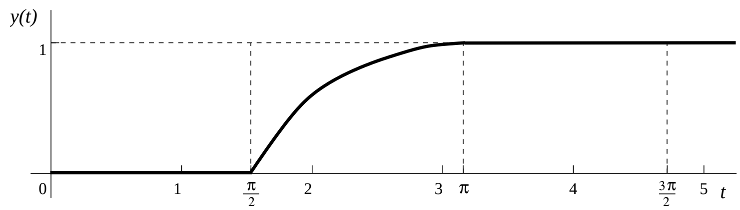
(20) 5. a) Solve the initial value problem $y'' + y = H(t-\pi) + \delta(t - \frac{\pi}{2})$ with $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$.

Answer Laplace transform gives $s^2Y(s) + Y(s) = \frac{e^{\pi s}}{s} + e^{\frac{\pi}{2}s}$. Therefore $Y(s) = \frac{e^{\pi s}}{(s^2+1)s} + \frac{e^{\frac{\pi}{2}s}}{s^2+1}$. The inverse Laplace transform of the second term can be read off the table: $H(t-\frac{\pi}{2})\sin(t-\frac{\pi}{2})$. The other term is an exponential multiplying $\frac{1}{(s^2+1)s}$. That needs splitting up by partial fractions: $\frac{As+B}{s^2+1} + \frac{C}{s}$. Then $s(As+B) + C(s^2+1) = 1$ so ($s=0$) $C = 1$ and $B = 0$ (s coefficient) and $A = -1$ (s^2 coefficient). Thus $\frac{1}{(s^2+1)s} = \frac{-s}{s^2+1} + \frac{1}{s}$ which maybe we could have guessed. Now we still want the inverse Laplace transform of $e^{\pi s}\left(\frac{-s}{s^2+1} + \frac{1}{s}\right)$. We can read this off the table: $H(t-\pi)\left(-\cos(t-\pi)+1\right)$. The complete answer is $y(t) = H(t-\frac{\pi}{2})\sin(t-\frac{\pi}{2}) + H(t-\pi)\left(-\cos(t-\pi)+1\right)$.

b) Write formulas *without* Heaviside functions for $y(t)$ in the indicated intervals:

Answer If $0 < t < \frac{\pi}{2}$ then $y(t) = 0$.
If $\frac{\pi}{2} < t < \pi$ then $y(t) = \sin(t-\frac{\pi}{2})$.
If $\pi < t$ then $y(t) = \sin(t-\frac{\pi}{2}) - \cos(t-\pi) + 1$ (a fine answer!). Amazingly, this is just $y(t) = 1$.

c) Graph $y(t)$ as well as you can on the axes below.



d) For which t in the interval $0 < t < 5$ is $y(t)$ differentiable?

Answer Certainly $y(t)$ is differentiable away from $t = \frac{\pi}{2}$ and $t = \pi$. In fact, we are lucky because $\sin(t-\frac{\pi}{2})$ has a maximum at π, so that the function is also differentiable at π.

(14) 6. a) Find the Laplace transform of this linear first-order system of ordinary differential equations $\begin{cases} 2x'(t) + 3x(t) + y'(t) = e^t \\ x'(t) + x(t) - y'(t) + 5y(t) = \sin t \end{cases}$ with initial conditions $\begin{cases} x(0) = 1 \\ y(0) = 0 \end{cases}$.

Answer $\begin{cases} 2sX(s) - 2 + 3X(s) + sY(s) = \frac{1}{s-1} \\ sX(s) - 1 + X(s) - sY(s) + 5Y(s) = \frac{1}{s^2+1} \end{cases}$

b) Find an expression for the Laplace transform, $X(s)$, of $x(t)$ which does not involve the Laplace transforms of $y(t)$. Do **not** simplify your answer! Do **not** try to compute $x(t)$!

Answer Rewrite the equations slightly: $\begin{cases} (2s+3)X(s) & +sY(s) & = & \frac{1}{s-1}+2 \\ (s+1)X(s) & +(5-s)Y(s) & = & \frac{1}{s^2+1}+1 \end{cases}$. Now we see two linear equations in two unknowns. Multiply the first equation by $5-s$ and the second equation by s and subtract: $\left((5-s)(2s+3) - s(s+1)\right)X(s) = (5-s)\left(\frac{1}{s-1}+2\right) - s\left(\frac{1}{s^2+1}+1\right)$ so that we can solve for $X(s) = \frac{(5-s)\left(\frac{1}{s-1}+2\right) - s\left(\frac{1}{s^2+1}+1\right)}{\left((5-s)(2s+3) - s(s+1)\right)}$.

Comment `Maple` reported in about .1 second that the inverse Laplace transform of $X(s)$ given above is

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2/9*exp(t)-1/20*cos(t)-1/60*sin(t)+(149/360-61/360*6^(1/2))*exp((6^(1/2)+1)*t)+(149/360+61/360*6^(1/2))*exp(-(6^(1/2)-1)*t)
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(16) 7. Show that $\mathbf{u} = (1,-1,0,1,1)$ and $\mathbf{v} = (2,2,-2,2,2)$ and $\mathbf{w} = (1,5,-3,1,1)$ are linearly dependent in $\mathbb{R}^5$.

Answer We need to find a non-trivial solution to the vector equation $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}$, which is $a(1,-1,0,1,1) + b(2,2,-2,2,2) + c(1,5,-3,1,1) = (0,0,0,0,0)$. We must solve the scalar system

$$\begin{cases} 1a + 2b + c = 0 \\ -1a + 2b + 5c = 0 \\ 0a - 2b - 3c = 0 \\ 1a + 2b + 1c = 0 \\ 1a + 2b + 1c = 0 \end{cases} \Longrightarrow \begin{cases} 1a + 2b + c = 0 \\ 0a + 4b + 6c = 0 \\ 0a - 2b - 3c = 0 \\ 0a + 0b + 0c = 0 \\ 0a + 0b + 0c = 0 \end{cases} \Longrightarrow \begin{cases} 1a + 0b - 2c = 0 \\ 0a + 1b + \frac{3}{2}c = 0 \\ 0a + 0b + 0c = 0 \\ 0a + 0b + 0c = 0 \\ 0a + 0b + 0c = 0 \end{cases}$$

The values $c = 1$ and $a = 2$ and $b = -\frac{3}{2}$ provide a non-trivial solution.