**640:421:02** **Answers to the Second Exam** **4/11/2004**

(12) 1. Complete the definitions.

a) Suppose $v_1$, $v_2$, $\ldots$ and $v_t$ are vectors in $\mathbb{R}^n$. Then a *linear combination* of $v_1$, $v_2$, $\ldots$ and $v_t$ is *any sum,* $\sum_{j=1}^{t} a_j v_j$*, where the* $a_j$*'s are scalars.*

b) Suppose $A$ is an $n$ by $n$ matrix. $\lambda$ is an *eigenvalue* of $A$ if *there is a non-zero vector* $X$ *in* $\mathbb{R}^n$ *so that (writing* $X$ *as a column vector)* $AX = \lambda X$ or if $\lambda$ *is a root of* $\det(A - \lambda I_n) = 0$.

(18) 2. Suppose that $A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$.

a) Compute the characteristic polynomial of $A$.

**Answer** $\det \begin{pmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{pmatrix} = (4-\lambda)(1-\lambda) + 2 = \lambda^2 - 5\lambda + 6$.

b) Find a basis of $\mathbb{R}^2$ consisting of eigenvectors of $A$.

**Answer** Since $\lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2)$, the eigenvalues are 3 and 2.

For $\lambda = 3$: $\begin{pmatrix} 4-3 & 1 \\ -2 & 1-3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}$. $(1, -1)$ is an eigenvector since $\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

For $\lambda = 2$: $\begin{pmatrix} 4-2 & 1 \\ -2 & 1-2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$. $(1, -2)$ is an eigenvector since $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The basis consists of the two vectors $(1, -1)$ and $(1, -2)$.

c) Find a diagonal matrix $D$ and an invertible matrix $P$ so that $P^{-1}AP = D$.

**Answer** $P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$ and $D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

d) Find $P^{-1}$. **Answer** $\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right)$ so $P^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

e) Compute $Z = AP$. **Answer** $Z = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix}$.

f) Compute $P^{-1}Z$. **Answer** $\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -3 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

(12) 3. For which values of $x$ is the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ x^2 & 3 & 2 & 4 \\ x & 4 & 2 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix}$ invertible?

**Answer** $A$ is invertible when $\det(A)$ is not zero. We can compute $\det(A)$ directly, but row and column operations don't change the determinant. So $\det(A) =$ (subtract the third column from the fourth) $\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ x^2 & 3 & 2 & 2 \\ x & 4 & 2 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} =$ (add the fourth column to all the others) $\det \begin{pmatrix} 1 & 0 & 1 & 0 \\ x^2+2 & 5 & 4 & 2 \\ x+1 & 5 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} =$ (expand along the fourth row) $-1 \det \begin{pmatrix} 1 & 0 & 1 \\ x^2+2 & 5 & 4 \\ x+1 & 5 & 3 \end{pmatrix} =$ (the 3 by 3 formula) $-\Big(\big(15 + 5(x^2+2)\big) - \big(20 + 5(x+1)\big)\Big) = -5x^2 + 5x$. This is 0 when $x$ is 0 or 1. For all other values of $x$, the matrix is invertible.

(14) 4. Suppose the vectors $(1, 0, 3, 2, -2)$ and $(-1, 1, 2, 2, 5)$ are both solutions to a homogeneous system, $AX = 0$, where $A$ is a 5 by 5 matrix, to be used in all parts of this problem.

a) Are there other solutions to this homogeneous system? What can be concluded about the dimension of the collection of all solutions, $S$, of this homogeneous system? **Answer** There are other solutions: any linear combination of $(1, 0, 3, 2, -2)$ and $(-1, 1, 2, 2, 5)$ is one. $S$ is a subspace of $\mathbb{R}^5$ containing the span of $(1, 0, 3, 2, -2)$ and $(-1, 1, 2, 2, 5)$. These vectors aren't multiples of each other so dim $S$ is at least 2.

b) Estimate the rank of $A$. **Answer** Since dim $S$ is at least 2, the "junk" in the RREF of A must be at least two columns wide. So the rank of $A$ is at most 3: it is 0 or 1 or 2 or 3.

**Note** The larger that $\dim S$ is, the smaller rank $A$ must be. In fact, here $\dim S + \operatorname{rank} A = 5$.

c) Consider the inhomogeneous system, $AX = Y$. Does this system have solutions for all $Y$'s in $\mathbb{R}^5$?

**Answer** There are certainly $Y$'s for which $AX = Y$ has no solution. The two bottom rows of the RREF of $A$ must be 0, since the rank of $A$ is at most 3. Therefore there are linear combinations of the $Y$ entries which *must* be 0 for the system $AX = Y$ to have a solution.

(16) 5. The matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ has eigenvalues 1 and 2 and 3, with associated eigenvectors $(-1,1,0)$ and $(0,0,1)$ and $(1,1,0)$. Use this information together with diagonalization to compute $A^5$.

**Note** $1^5 = 1$ and $2^5 = 32$ and $3^5 = 243$. The entries in the answer are 0, 32, 121, or 122.

**Answer** If $B = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then $B^{-1}AB = D$. Let's get $B^{-1}$:

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right)$$

Then $A^5 = B\left(D^5\right)B^{-1}$. Since $D^5 = \begin{pmatrix} 1^5 & 0 & 0 \\ 0 & 2^5 & 0 \\ 0 & 0 & 3^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{pmatrix}$, we compute $\left(D^5\right)B^{-1} =$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 243 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 32 \\ \frac{243}{2} & \frac{243}{2} & 0 \end{pmatrix}$. Multiply on the left by $B$ to get $\begin{pmatrix} 122 & 121 & 0 \\ 121 & 122 & 0 \\ 0 & 0 & 32 \end{pmatrix}$.

(16) 6. **BIRD** Suppose $\mathbf{u} = (2,3,1,5,1)$ and $\mathbf{v} = (0,5,3,1,-1)$ and $\mathbf{w} = (2,6,-2,-2,5)$.

a) Show that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent in $\mathbb{R}^5$.

**Answer** Use **GOLDFINCH** (or **NUTHATCH**), which has the vectors as row vectors (column vectors, respectively). Since the rank is 3, the three vectors are linearly independent.

b) Find a vector in $\mathbb{R}^5$ which is *not* a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Verify your answer. **Answer** I'll guess that $(1,1,1,1,1)$ is not a linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Then $m\mathbf{u}+n\mathbf{v}+p\mathbf{w} = (1,1,1,1,1)$ should have no solution. Written out as a system in matrix form, this is $\begin{pmatrix} 2 & 0 & 2 \\ 3 & 5 & 6 \\ 1 & 3 & -2 \\ 5 & 1 & -2 \\ 1 & -1 & 5 \end{pmatrix}\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. But

$$\left(\begin{array}{ccc|c} 2 & 0 & 2 & a \\ 3 & 5 & 6 & b \\ 1 & 3 & -2 & c \\ 5 & 1 & -2 & d \\ 1 & -1 & 5 & e \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{7}{12}a - \frac{1}{8}b + \frac{5}{24}c \\ 0 & 1 & 0 & -\frac{1}{4}a + \frac{1}{8}b + \frac{1}{8}c \\ 0 & 0 & 1 & \frac{1}{12}a + \frac{1}{8}b - \frac{5}{24}c \\ 0 & 0 & 0 & -\frac{17}{6}a + \frac{3}{4}b - \frac{19}{12}c + d \\ 0 & 0 & 0 & -\frac{5}{12}a - \frac{3}{8}b + \frac{23}{24}c + e \end{array}\right)$$

This is **NUTHATCH**. If $a = b = c = d = e = 1$ then the last row represents the equation $0 = -\frac{5}{12} - \frac{3}{8} + \frac{23}{24} + 1$. The right side is bigger than 1, and certainly isn't 0. So this system is inconsistent, and $(1,1,1,1,1)$ is not in the span of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

(12) 7. **BIRD** Use linear algebra to find a polynomial $p(x)$ of degree 3 so that $p(1) = 2$, $p'(1) = -2$, $p(2) = 2$, and $p'(2) = -2$.

**Answer** If $p(x) = A + Bx + Cx^2 + Dx^3$, then $p'(x) = B + 2Cx + 3Dx^2$. The conditions given translate to

$$\begin{cases} 1A + 1B + 1C + \phantom{1}1D = \phantom{-}2 \\ 0A + 1B + 2C + \phantom{1}3D = -2 \\ 1A + 2B + 4C + \phantom{1}8D = \phantom{-}2 \\ 0A + 1B + 4C + 12D = -2 \end{cases}$$

This is a linear system. It is time to quote **HELICOPTER**:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & a \\ 0 & 1 & 2 & 3 & b \\ 1 & 2 & 4 & 8 & c \\ 0 & 1 & 4 & 12 & d \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4a - 4b + 5c - 2d \\ 0 & 1 & 0 & 0 & 12a + 8b - 12c + 5d \\ 0 & 0 & 1 & 0 & -9a - 5b + 9c - 4d \\ 0 & 0 & 0 & 1 & 2a + b - 2c + d \end{array}\right)$$

If $a = 2$, $b = -2$, $c = 2$, and $d = -2$, then $-4a - 4b + 5c - 2d = -8 + 8 + 10 + 4 = 14$, $12a + 8b - 12c + 5d = 24 - 16 - 24 - 10 = -26$, $-9a - 5b + 9c - 4d = -18 + 10 + 18 + 8 = 18$, and $2a + b - 2c + d = 4 - 2 - 4 - 2 = -4$. The polynomial is $p(x) = 14 - 26x + 18x^2 - 4x^3$. You can check that this is correct – I did!